第四章 波动方程

考虑一阶偏微分方程

$$\begin{cases} \dfrac{\partial u}{\partial t} + a(x,t)\dfrac{\partial u}{\partial x} + b(x,t)\,u = f(x,t) \qquad -\infty < x < +\infty \\ u(x,0) = \phi(x). \end{cases}$$

$u(x,t)$

特征线法：

$$\begin{cases} \dfrac{dx(t)}{dt} = a(x(t),t). \\ x(0) = c \end{cases}$$

令 $U(t) = u(x(t),t)$，则

$$\frac{dU}{dt} = \frac{dx(t)}{dt}(\partial_x u)(x(t),t) + \frac{\partial u}{\partial t}(x(t),t)$$

$$= \big(a(x,t)\partial_x u\big)(x(t),t) + \frac{\partial u}{\partial t}(x(t),t).$$

$$\Longrightarrow \begin{cases} \dfrac{dU}{dt} = -\,b(x(t),t)U(t) + f(x(t),t). \\ U(0) = \phi(c). \end{cases}$$

例1. $\begin{cases} \dfrac{\partial u}{\partial t} - a\dfrac{\partial u}{\partial x} = f(x,t) \\ u(x,0) = \phi(x) \end{cases}$ a是常数.

特征线: $\begin{cases} \dfrac{dx(t)}{dt} = -a \\ x(0) = c \end{cases} \Longrightarrow x(t) = -at + c$

令 $U(t) = u(c-at,t)$，则

$$\begin{cases} \dfrac{dU}{dt} = \left(\dfrac{\partial u}{\partial t} - a\dfrac{\partial u}{\partial x}\right)(c-at,t) = f(c-at,t). \\ U(0) = \phi(c) \end{cases}$$

$$\Longrightarrow U(t) - \phi(c) = \int_0^t f(c-as,s)\,ds.$$

$$\Longrightarrow \underset{\substack{\| \\ u(\underbrace{c-at}_{x},\,t).}}{U(t)} = \phi(c) + \int_0^t f(c-as,s)\,ds$$

$$\Longrightarrow u(x,t) = \phi(x+at) + \int_0^t f(x+at-as,s)\,ds$$

$$= \phi(x+at) + \int_0^t f(x+a(t-s),s)\,ds.$$

例12. $\begin{cases} \dfrac{\partial u}{\partial t} + (x+t)\dfrac{\partial u}{\partial x} + u = x \\ u(x,0) = x \end{cases}$

特征线: $\begin{cases} \dfrac{dx(t)}{dt} = x(t) + t \\ x(0) = c \end{cases}$

$\Longrightarrow x(t) = ce^{t} + e^{t} - t - 1$

令 $U(t) = u(ce^{t}+e^{t}-t-1,\ t)$, 则

$$\begin{cases} \dfrac{dU}{dt} = \left(\dfrac{\partial u}{\partial t} - (x+t)\dfrac{\partial u}{\partial x}\right)(ce^{t}+e^{t}-t-1,\ t) = -U(t) + ce^{t} + e^{t} - t - 1 \\ U(0) = c \end{cases}$$

$$\Longrightarrow \underset{\underset{u(\underbrace{ce^{t}+e^{t}-t-1}_{x},\ t)}{\parallel}}{U(t)} = -t + \frac{1}{2}(e^{t} - e^{-t}) + \frac{c}{2}(e^{t} - e^{-t}).$$

$c = (x - e^{t} + t + 1)e^{-t} = xe^{-t} - 1 + te^{-t} + e^{-t}$

$$\Longrightarrow u(x,t) = \frac{1}{2}(x-t+1) - e^{-t} + \frac{1}{2}(x+t+1)e^{-2t}.$$

波动方程. $\partial_t^2 u - \Delta u = f(x,t) \qquad t \in \mathbb{R},\ x \in \Omega \subset \mathbb{R}^n$

初值: $u(x,0) = \varphi(x),\quad \partial_t u(x,0) = \psi(x)$.

边值: Dirichlet边值: $u(x,t) = g(x,t),\ \forall x \in \partial\Omega,\ t \in I$ 第I类

Neumann边值 $\dfrac{\partial u}{\partial n}(x,t) = g(x,t),\ \forall x \in \partial\Omega,\ t \in I.$ 第II类

Robin边值 $\dfrac{\partial u}{\partial n} + \alpha(x,t)u = g(x,t),\ \forall x \in \partial\Omega,\ t \in I.$ 第III类.

§4.1. 初值问题

$$(*)\ \begin{cases} \partial_t^2 u - \Delta u = f(x,t). & x\in\mathbb{R}^n,\ t\in\mathbb{R} \\ u(x,0)=\varphi(x),\ \ \partial_t u(x,0)=\psi(x). \end{cases}$$

$$①:\ \begin{cases} \partial_t^2 u_1 - \Delta u_1 = 0 \\ u_1(x,0)=\varphi(x),\ \ \partial_t u_1(x,0)=0 \end{cases} \quad (4.7)$$

$$②:\ \begin{cases} \partial_t^2 u_2 - \Delta u_2 = 0 \\ u_2(x,0)=0,\ \ \partial_t u_2(x,0)=\psi(x). \end{cases} \quad (4.8)$$

$$③:\ \begin{cases} \partial_t^2 u_3 - \Delta u_3 = f(x,t) \\ u_3(x,0)=0,\ \ \partial_t u_3(x,0)=0. \end{cases} \quad (4.9).$$

则 $u=u_1+u_2+u_3$ 是(*)的解.

定理4.1. 设 $u_2=M_\psi(x,t)$ 是初值问题(4.8)的解，则初值问题(4.7)的解为 $u_1=\frac{\partial}{\partial t}M_\varphi$. 初值问题(4.9)的解为

$$u_3=\int_0^t M_{f_\tau}(x,t-\tau)\,d\tau.$$

证：令 $\tilde{u}(x,t)=M_\varphi(x,t)$. 则

$$\begin{cases} \partial_t^2 \tilde{u} - \Delta \tilde{u} = 0 \\ \tilde{u}(x,0)=0,\ \ \partial_t \tilde{u}(x,0)=\varphi(x). \end{cases}$$

对方程两边求关于t的偏导数，有（令 $v=\partial_t\tilde{u}$）

$$\begin{cases} \partial_t^2 v - \Delta v = 0. \\ v(x,0)=\varphi(x),\ \ \partial_t v(x,0)=0. \end{cases}$$

令 $\tilde{u}(x,t)=M_{f_\tau}(x,t)$. 则

$$\begin{cases} \partial_t^2 \tilde{u} - \Delta \tilde{u} = 0 \\ \tilde{u}(x,0)=0,\ \ \partial_t \tilde{u}(x,0)=f_\tau(x)=f(x,\tau) \end{cases}$$

令 $v(x,t)=\int_0^t M_{f_\tau}(x,t-\tau)\,d\tau$.

$$\partial_t v=\int_0^t (\partial_t\tilde{u})(t-\tau)\,d\tau$$

$$\partial_t^2 v=f(x,t)+\int_0^t (\partial_t^2\tilde{u})(t-\tau)\,d\tau.$$

$$\Delta v = \int_0^t (\Delta \tilde{u})(t-\tau)\,d\tau$$

$$\Longrightarrow \begin{cases} \partial_t^2 v - \Delta v = f(x,t) + \int_0^t (\partial_t^2 \tilde{u} - \Delta \tilde{u})(t-\tau)\,d\tau = f(x,t) \\ v(x,0)=0,\ \partial_t v(x,0) = 0 \end{cases}$$

一维初值问题 考虑直线上的波动方程

$$\begin{cases} \partial_t^2 u - \partial_x^2 u = f(x,t) \\ u(x,0)=\varphi(x),\ \partial_t u(x,0)=\psi(x) \end{cases} \quad (\Delta)$$

关键是求解 $\begin{cases} \partial_t^2 u - \partial_x^2 u = 0 \\ u(x,0)=0,\ \partial_t u(x,0)=\psi(x) \end{cases}$

$$(\partial_t+\partial_x)\underbrace{(\partial_t-\partial_x)u}_{v(x,t)} = (\partial_t^2-\partial_x^2)u = 0$$

令 $v(x,t)=\partial_t u - \partial_x u$，则

$$\begin{cases} \partial_t v + \partial_x v = 0 \\ v(x,0)=\psi(x) \end{cases}$$

特征线 $\begin{cases} \dfrac{dx(t)}{dt} = 1 \\ x(0)=c \end{cases} \Longrightarrow x(t)=t+c$

令 $U(t)=v(t+c,t)$，则 $\begin{cases} \dfrac{dU}{dt} = \dfrac{\partial v}{\partial t} + \dfrac{\partial v}{\partial x} = 0 \\ U(0)=\psi(c) \end{cases}$

$$\Longrightarrow U(t)=\psi(c)$$

$$\| $$

$$v(\underbrace{t+c}_{x},t)$$

$$\Longrightarrow v(x,t)=\psi(x-t)$$

再解 $\begin{cases} \partial_t u - \partial_x u = \psi(x-t) \\ u(x,0)=0 \end{cases}$

$$-\frac{1}{2}\int_{x+t}^{x-t} \psi(y)\,dy$$

特征线 $x(t)=-t+c$

令 $U(t)=u(c-t,t)$，则 $\begin{cases} \dfrac{dU}{dt} = \left(\dfrac{\partial u}{\partial t} - \dfrac{\partial u}{\partial x}\right)(c-t,t) = \psi(c-2t) \\ U(0)=0 \end{cases}$

$$\Longrightarrow U(t) = \int_0^t \psi(c-2s)\,ds$$

$$u(\underbrace{c-t}_{x},t) =$$

$$\Longrightarrow u(x,t) = \int_0^t \psi(x+t-2s)\,ds \overset{y=x+t-2s}{\underset{dy=-2ds}{=\!=}} \frac{1}{2}\int_{x-t}^{x+t} \psi(y)\,dy$$

由 Thm 4.1. $\begin{cases} \partial_t^2 u - \partial_x^2 u = 0 \\ u(x,0) = \varphi(x),\ \partial_t u(x,0) = 0 \end{cases}$ 的解为

$$u_1(x,t) = \frac{1}{2}\left(\varphi(x+t) + \varphi(x-t)\right)$$

$$u_3(x,t) = \frac{1}{2}\int_0^t \int_{x-(t-\tau)}^{x+(t-\tau)} f(y,\tau)\,dy\,d\tau.$$

故 (Δ) 的解为 $u(x,t) = \frac{1}{2}(\varphi(x+t) + \varphi(x-t)) + \frac{1}{2}\int_{x-t}^{x+t} \psi(y)\,dy + \frac{1}{2}\int_0^t \int_{x-(t-\tau)}^{x+(t-\tau)} f(y,\tau)\,dy\,d\tau$ (D'Alembert 公式).