$$\begin{cases}\partial_t^2 u-\Delta u=f(x,t) & x\in\mathbb{R}^n,\ t\in\mathbb{R}\\ u(x,0)=\varphi(x),\quad \partial_t u(x,0)=\psi(x)\end{cases}$$

$$u=u_1+u_2+u_3$$

$$\begin{cases}\partial_t^2 u_2-\Delta u_2=0\\ u_2(x,0)=0,\ \partial_t u_2(x,0)=\psi(x).\end{cases}$$

$$\begin{cases}\partial_t^2 u_3-\Delta u_3=f(x,t)\\ u_3(x,0)=0,\ \partial_t u_3(x,0)=0.\end{cases}$$

$$t=t_i\quad\begin{cases}\partial_t^2 v-\Delta v=0\\ v(x,t_i)=0,\ \partial_t v(x,t_i)=f_{t_i}\end{cases}$$

$$v\Delta t_i=\tilde v\quad\begin{cases}\partial_t^2 \tilde v-\Delta \tilde v=0\\ \tilde v(x,t_i)=0,\ \partial_t \tilde v(x,t_i)=f_{t_i}\Delta t_i\end{cases}$$

$$u_2(x,t)=M_\psi(x,t).$$

$$u_3(x,t)=\int_0^t M_{f_\tau}(x,t-\tau)\,d\tau$$

$$=\lim_{\lambda\to0}\sum_{i=1}^N \underbrace{M_{f_{t_i}}(x,t-t_i)}_{v}\Delta t_i$$

$$=\lim_{\lambda\to0}\sum_{i=1}^N M_{f_{t_i}\Delta t_i}(x,t-t_i).$$

$$0=t_0<t_1<\cdots<t_N=t.$$

$$\lambda\triangleq\max_{1\le i\le N}|t_i-t_{i-1}|\to0.$$

冲量定理 齐次化. | Duhamel 原理.

不变性：方程 $\partial_t^2 u-\Delta u=0$ 在以下变换下不变.

① 时间平移 $u(x,t)\mapsto u(x,t+t_0)$, $\forall t_0\in\mathbb{R}$ 能量.

② 空间平移 $u(x,t)\mapsto u(x+x_0,t)$, $\forall x_0\in\mathbb{R}^n$. 动量.

③ 伸缩： $u(x,t)\mapsto u(\frac{x}{\lambda},\frac{t}{\lambda})$, $\forall\lambda>0$. Virial恒等式.

④ 洛伦兹： $u(x,t)\mapsto u\left(\frac{t-v\cdot x}{\sqrt{1-|v|^2}},\ x-x_v-\frac{x_v-vt}{\sqrt{1-|v|^2}}\right)$, $x_v=\left(x\cdot\frac{v}{|v|}\right)\frac{v}{|v|}$

$v\in\mathbb{R}^n$, $|v|<1$.

$n=1$

$$\begin{cases}\partial_t^2 u-\partial_x^2 u=f(x,t), & x\in\mathbb{R},\ t\in\mathbb{R}\\ u(x,0)=\varphi(x),\ \partial_t u(x,0)=\psi(x)\end{cases}\qquad(4.13).$$

D'Alembert公式：

$$u(x,t)=\frac12\big(\varphi(x+t)+\varphi(x-t)\big)+\frac12\int_{x-t}^{x+t}\psi(y)\,dy+\frac12\int_0^t\int_{x-(t-\tau)}^{x+(t-\tau)}f(y,\tau)\,dy\,d\tau.\qquad(4.20)$$

$f\equiv 0$. 令 $F(x)=\frac{1}{2}\varphi(x)+\frac{1}{2}\int_0^x\psi(y)dy$, $G(x)=\frac{1}{2}\varphi(x)-\frac{1}{2}\int_0^x\psi(y)dy$.

则 $u(x,t)=F(x+t)+G(x-t)$. (左行解、右行解)

定理4.2. 若 $\varphi\in C^2(\mathbb{R})$, $\psi\in C^1(\mathbb{R})$, $f\in C^1(\mathbb{R}\times\mathbb{R})$

则由表达式(4.20)给出的函数 $u\in C^2(\mathbb{R}\times\mathbb{R})$

是初值问题(4.13)的解(古典解).

推论4.3. 若 φ, ψ 及 f 是 x 的偶(或奇、或周期为 L 的)函数. 则由(4.20)给出的解也必然是 x 的偶(或奇、或周期为 L 的)函数.

一维半无界问题
$$\begin{cases} u_{tt}-u_{xx}=f(x,t), & x>0,\ t\in\mathbb{R}. \\ u(x,0)=\varphi(x),\ \partial_t u(x,0)=\psi(x) & (4.21) \\ u(0,t)=g(t) \end{cases}$$

☆ $g(t)\equiv 0$.

令 $\bar\varphi(x)=\begin{cases}\varphi(x) & x\geq 0\\ -\varphi(-x) & x<0\end{cases}$, $\bar\psi(x)=\begin{cases}\psi(x) & x\geq 0\\ -\psi(-x) & x<0,\end{cases}$

$\bar f(x,t)=\begin{cases} f(x,t) & x\geq 0\\ -f(-x,t), & x<0\end{cases}$

令 $\bar u(x,t)$ 是
$$\begin{cases}\partial_t^2\bar u-\partial_x^2\bar u=\bar f(x,t), & x\in\mathbb{R},\ t\in\mathbb{R}. \\ \bar u(x,0)=\bar\varphi(x),\ \partial_t\bar u(x,0)=\bar\psi(x)\end{cases}$$

由推论4.3, $\bar u(x,t)$ 是奇函数. 故 $\bar u(0,t)=0$

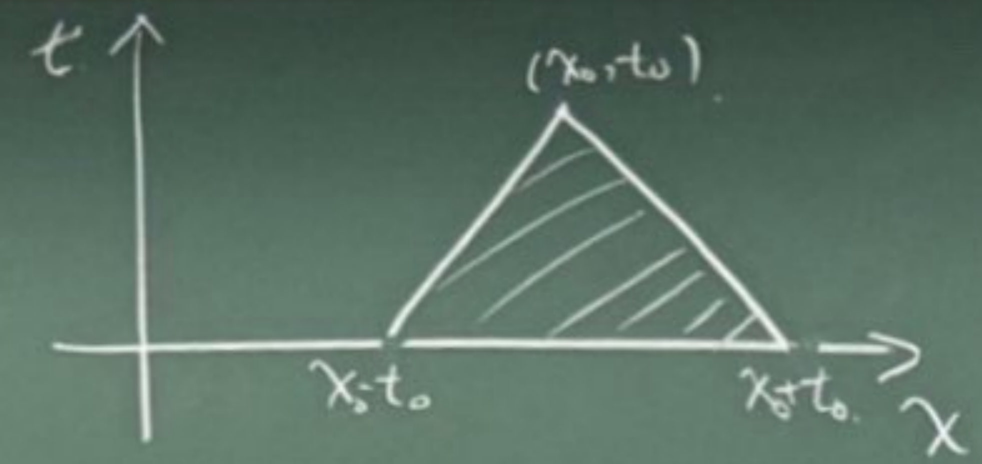
由D'Alembert公式,

$$\bar u(x,t)=\frac{1}{2}\left(\bar\varphi(x+t)+\bar\varphi(x-t)\right)+\frac{1}{2}\int_{x-t}^{x+t}\bar\psi(y)dy+\frac{1}{2}\int_0^t\int_{x-(t-\tau)}^{x+(t-\tau)}\bar f(y,\tau)dyd\tau$$

下面只考虑 $t\geq 0$.

当 $x\ge t$ 时,

$u(x_0,t_0)$



$$u(x,t)=\frac{1}{2}\big(\varphi(x+t)+\varphi(x-t)\big)+\frac{1}{2}\int_{x-t}^{x+t}\psi(y)dy+\frac{1}{2}\int_0^t\int_{x-(t-\tau)}^{x+(t-\tau)}f(y,\tau)\,dy\,d\tau \qquad (4.23)$$

当 $x<t$ 时,

$$u(x,t)=\frac{1}{2}\big(\varphi(x+t)-\varphi(t-x)\big)+\frac{1}{2}\int_{t-x}^{x+t}\psi(y)dy+\frac{1}{2}\int_0^{t-x}\left(\int_{x-(t-\tau)}^{0}-f(-y,\tau)dy+\int_0^{x+(t-\tau)}f(y,\tau)dy\right)d\tau$$

$$+\frac{1}{2}\int_{t-x}^{t}\int_{x-(t-\tau)}^{x+(t-\tau)}f(y,\tau)\,dy\,d\tau$$

$$=\frac{1}{2}\big(\varphi(x+t)-\varphi(t-x)\big)+\frac{1}{2}\int_{t-x}^{x+t}\psi(y)dy+\frac{1}{2}\int_{t-x}^{t}\int_{x-(t-\tau)}^{x+(t-\tau)}f(y,\tau)d\tau+\frac{1}{2}\int_0^{t-x}\int_{t-\tau-x}^{(t-\tau)+x}f(y,\tau)\,dy\,d\tau \qquad (4.24)$$

★ $g(t)$ 不恒为零.

令 $v(x,t)=u(x,t)-g(t)$, 则

$$\begin{cases}\partial_t^2 v-\partial_x^2 v=f(x,t)-g''(t), & x>0,\ t\in R\\ v(x,0)=\varphi(x)-g(0), \quad \partial_t v(x,0)=\psi(x)-g'(0)\\ v(0,t)=0\end{cases}$$

相容性条件: $g(0)=\varphi(0)$, $g'(0)=\psi(0)$, $g''(0)=\varphi''(0)+f(0,0)$. (*)

定理4.3. $u\in C^2(\bar{R}_+\times R)$

定理 4.3 若半无界问题 (4.21) 的初值 $\varphi(x)\in C^2(\bar{\mathbf{R}}_+)$, $\psi(x)\in C^1(\bar{\mathbf{R}}_+)$ 及非齐次项 $f(x,t)\in C^1(\bar{\mathbf{R}}_+\times\bar{\mathbf{R}}_+)$ 满足相容性条件 (4.25), (4.26) 和 (4.27), 且边值 $g(t)\equiv 0$, 则由公式 (4.23) 和 (4.24) 给出的函数 $u\in C^2(\bar{\mathbf{R}}_+\times\bar{\mathbf{R}}_+)$, 且是半无界问题 (4.21) 的解.

高维初值问题

$n=3.$ $\begin{cases} \partial_t^2 u - \Delta_{\mathbb{R}^3} u = f(x,t) & x \in \mathbb{R}^3, t \in \mathbb{R}. \\ u(x,0) = \varphi(x), & \partial_t u(x,0) = \psi(x) \end{cases}$

由Thm 4.1, 先考虑 $\begin{cases} \partial_t^2 u - \Delta_{\mathbb{R}^3} u = 0 \\ u(x,0) = 0, \ \partial_t u(x,0) = \psi(x). \end{cases}$

(r, θ, φ)

球面平均法

$$\Delta = \partial_r^2 + \frac{2}{r}\partial_r + \frac{1}{r^2}\Delta_{S^2}$$

$$\partial_t^2 u - \partial_r^2 u - \frac{2}{r}\partial_r u - \frac{1}{r^2}\Delta_{S^2} u = 0$$

$$\int_\Omega \operatorname{div} \vec{F}\, dx = \int_{\partial\Omega} \vec{F} \cdot \vec{n}\, dS$$

$$\Delta_{S^2} = \operatorname{div}_{S^2} \nabla_{S^2}$$

$$\Rightarrow \int_{S^2} \Delta_{S^2} u\, d\omega = \int_{S^2} \operatorname{div}_{S^2}(\nabla_{S^2} u)\, d\omega = 0$$

在S^2上求平均, 令 $\bar{u}(t,r) = \frac{1}{4\pi}\int_{S^2} u(t, r\omega)\, d\omega.$

则 $\partial_t^2 \bar{u} - \left(\partial_r^2 \bar{u} + \frac{2}{r}\partial_r \bar{u}\right) = 0.$, $r > 0$

令 $\bar{u}(t,r) = r^k v(t,r).$ $\quad v(t,r) = \bar{u}(t, r^{u}).$

则 $\partial_t^2 \bar{u} = r^k \partial_t^2 v$

$\partial_r \bar{u} = k r^{k-1} v + r^k \partial_r v.$

$\partial_r^2 \bar{u} = k(k-1) r^{k-2} v + 2k r^{k-1} \partial_r v + r^k \partial_r^2 v$

$$r^k \partial_t^2 v - \left(\underline{k(k-1) r^{k-2} v} + 2k\, r^{k-1} \partial_r v + r^k \partial_r^2 v + \underline{2k\, r^{k-2} v} + 2 r^{k-1} \partial_r v \right) = 0.$$

$$\Rightarrow \partial_t^2 v - \left(k\underline{(k+1)} r^{-2} v + 2\underline{(k+1)} r^{-1} \partial_r v + \partial_r^2 v \right) = 0.$$

$k = -1$. $\mathbb{R}^1 \setminus$ $\quad v = r\,\bar{u}(t,r)$. $\mathbb{R}^1 \setminus$

$$\begin{cases} \partial_t^2 v - \partial_r^2 v = 0. \\ v(r,0) = \cdots \quad \partial_t v(r,0) = \cdots \end{cases}$$