$\mathbb{R}^n$上波方程

$n=1$.
$$\begin{cases}\partial_t^2 u-\partial_x^2 u=f(x,t)\\ u(x,0)=\varphi(x),\ \partial_t u(x,0)=\psi(x)\end{cases}$$

$$u(x,t)=\frac12\big(\varphi(x+t)+\varphi(x-t)\big)+\frac12\int_{x-t}^{x+t}\psi(y)dy+\frac12\int_0^t\int_{x-(t-\tau)}^{x+(t-\tau)}f(y,\tau)dyd\tau.$$
D'Alembert公式

$n=3$.
$$\begin{cases}\partial_t^2 u-\Delta_{\mathbb{R}^3}u=0\\ u(x,0)=0,\ \partial_t u(x,0)=\psi(x).\end{cases}$$

令 $\bar u(t,r)=\frac{1}{4\pi}\int_{S^2}u(x,r\omega)\,dS(\omega)$.

则
$$\begin{cases}\partial_t^2\bar u-\partial_r^2\bar u-\frac{2}{r}\partial_r\bar u=0,\quad r>0.\\ \bar u(r,0)=0\quad \partial_t\bar u(r,0)=\bar\psi(r)=\frac{1}{4\pi}\int_{S^2}\psi(r\omega)dS(\omega)\end{cases}$$

令 $v(t,r)=r\bar u(r,t)$. 则
$$\begin{cases}\partial_t^2 v-\partial_r^2 v=0\\ v(r,0)=0,\ \partial_t v(r,0)=r\bar\psi(r).\end{cases}\quad r>0.\quad (*)$$

用偶延拓，可将(*)变为$\mathbb{R}$上波动方程. 将延拓之后的解仍记为v.

Step1. $u(0,t)=\bar u(0,t)=\partial_r(r\bar u)\big|_{r=0}$

$=\frac12\Big[(r+t)\bar\psi(r+t)-(r-t)\bar\psi(r-t)\Big]\Big|_{r=0}$ $\quad y=x_0+t\omega$

$=\frac12\big(t\bar\psi(t)+t\bar\psi(-t)\big)$. $\quad dS(y)=t^2dS(\omega)\quad |\omega|=\left|\frac{y-x_0}{t}\right|=1.$

$=t\bar\psi(t)$.

Step 2. 对任意的 $x_0\in\mathbb{R}^3$, $\tilde u(x,t)=u(\cdot+x_0,t)$ 仍然是

$$\partial_t^2\tilde u-\Delta_{\mathbb{R}^3}\tilde u=0.$$
$$\tilde u(x,0)=0,\ \partial_t\tilde u(x,0)=\psi(x+x_0)\triangleq\tilde\psi(x).$$

由Step1. $\tilde u(0,t)=t\bar{\tilde\psi}(t)$

$$\tilde u(0,t)=u(x_0,t),\qquad t\bar{\tilde\psi}(t)=t\frac{1}{4\pi}\int_{S^2}\tilde\psi(t\omega)\,dS(\omega)$$

$$\frac{1}{4\pi t}\int_{|y-x_0|=t}\psi(y)\,dS(y)=\frac{t}{4\pi}\int_{|\omega|=1}\psi(x_0+t\omega)\,dS(\omega).$$

即 $\forall x_0 \in R^3$，$u(x_0,t)=\frac{1}{4\pi t}\int_{|y-x_0|=t}\psi(y)\,dS(y)$.

故 $u(x,t)=\frac{1}{4\pi t}\int_{|y-x|=t}\psi(y)\,dS(y)$.

由Thm4.1，$\begin{cases}\partial_t^2 u-\Delta_{R^3}u=f(x,t)\\ u(x,0)=\varphi(x),\ \partial_t u(x,0)=\psi(x).\end{cases}$

的解为

$$u(x,t)=\frac{\partial}{\partial t}\left(\frac{1}{4\pi t}\int_{|x-y|=t}\varphi(y)\,dS(y)\right)+\frac{1}{4\pi t}\int_{|y-x|=t}\psi(y)\,dS(y)$$
$$+\int_0^t\frac{1}{4\pi(t-\tau)}\int_{|y-x|=t-\tau}f(y,\tau)\,dS(y)\,d\tau.$$

(Kirchhoff公式).

$n=2$ 升维法.

$u(x_1,x_2,t)$：$\begin{cases}\partial_t^2 u-\Delta_{R^2}u=f(x,t)\\ u(x,0)=\varphi(x),\ \partial_t u(x,0)=\psi(x)\end{cases}$

令 $\tilde{u}(x_1,x_2,x_3,t)=u(x_1,x_2,t)$.

则 $\begin{cases}\partial_t^2\tilde{u}-\Delta_{R^3}\tilde{u}=\tilde{f}(\tilde{x},t). & \tilde{x}=(x_1,x_2,x_3).\\ \tilde{u}(\tilde{x},0)=\tilde{\varphi}(\tilde{x}),\ \partial_t\tilde{u}(\tilde{x},0)=\tilde{\psi}(\tilde{x}).\end{cases}$

$\tilde{x}=(x_1,x_2,x_3)$ $\tilde{\varphi}(\tilde{x})=\varphi(x_1,x_2)$，$\tilde{\psi}(\tilde{x})=\psi(x)$，$\tilde{f}(\tilde{x},t)=f(x,t)$.

先设 $\tilde{f}\equiv 0$，$\tilde{\varphi}\equiv 0$. 则由Kirchhoff公式，

$$\tilde{u}(\tilde{x},t)=\frac{1}{4\pi t}\int_{|\tilde{y}-\tilde{x}|=t}\tilde{\psi}(\tilde{y})\,dS(\tilde{y}).$$
$$\parallel$$
$$\tilde{u}(x_1,x_2,x_3,t).$$

由于 $\tilde{u}(\tilde{x},t)$ 与 x_3 无关，不妨设 $x_3=0$. 再令 $x_1=x_2=0$.

则 $u(0,t)=\tilde{u}(0,t)=\frac{1}{4\pi t}\int_{|\tilde{y}|=t}\tilde{\psi}(\tilde{y})\,dS(\tilde{y})$. $\quad y_1^2+y_2^2+y_3^2=t^2$

$$=\frac{2}{4\pi t}\int_{y_3=\sqrt{t^2-y_1^2-y_2^2}}\psi(y_1,y_2)\,dS(\tilde{y}).$$

$$= \frac{2}{4\pi t}\int_{y_1^2+y_2^2\le t^2} \psi(y_1,y_2)\frac{t}{\sqrt{t^2-y_1^2-y_2^2}}\,dy_1dy_2.$$

$$= \frac{1}{2\pi}\int_{|y|\le t}\frac{\psi(y_1,y_2)}{\sqrt{t^2-y_1^2-y_2^2}}\,dy_1dy_2.$$

$\forall\ x_0\in\mathbb{R}^2$, $v(x,t)=u(x+x_0,t)$. 则

$$\begin{cases}\partial_t^2 v-\Delta_{\mathbb{R}^2}v=0\\ v(x,0)=0,\ \partial_t v(x,0)=\psi(x+x_0).\end{cases}$$

于是，

$$\underset{u(x_0,t)}{\underset{\parallel}{v(0,t)}}=\frac{1}{2\pi}\int_{|y|\le t}\frac{\psi(x_0+y)}{\sqrt{t^2-|y|^2}}\,dy_1dy_2.$$

$$\overset{x_0+y\to y}{=\!=}\frac{1}{2\pi}\int_{|y-x_0|\le t}\frac{\psi(y)}{\sqrt{t^2-|y-x_0|^2}}\,dy_1dy_2$$

由 x_0 的任意性，

$$u(x,t)=\frac{1}{2\pi}\int_{|y-x|\le t}\frac{\psi(y)}{\sqrt{t^2-|y-x|^2}}\,dy_1dy_2.$$

再由 Thm 4.1,

$$u(x,t)=\frac{\partial}{\partial t}\left(\frac{1}{2\pi}\int_{|y-x|\le t}\frac{\varphi(y)}{\sqrt{t^2-|y-x|^2}}\,dy_1dy_2\right)+\frac{1}{2\pi}\int_{|y-x|\le t}\frac{\psi(y)}{\sqrt{t^2-|y-x|^2}}\,dy_1dy_2$$

$$+\int_0^t\frac{1}{2\pi}\int_{|y-x|\le t-\tau}\frac{f(y,\tau)}{\sqrt{(t-\tau)^2-|y-x|^2}}\,dy_1dy_2d\tau.$$

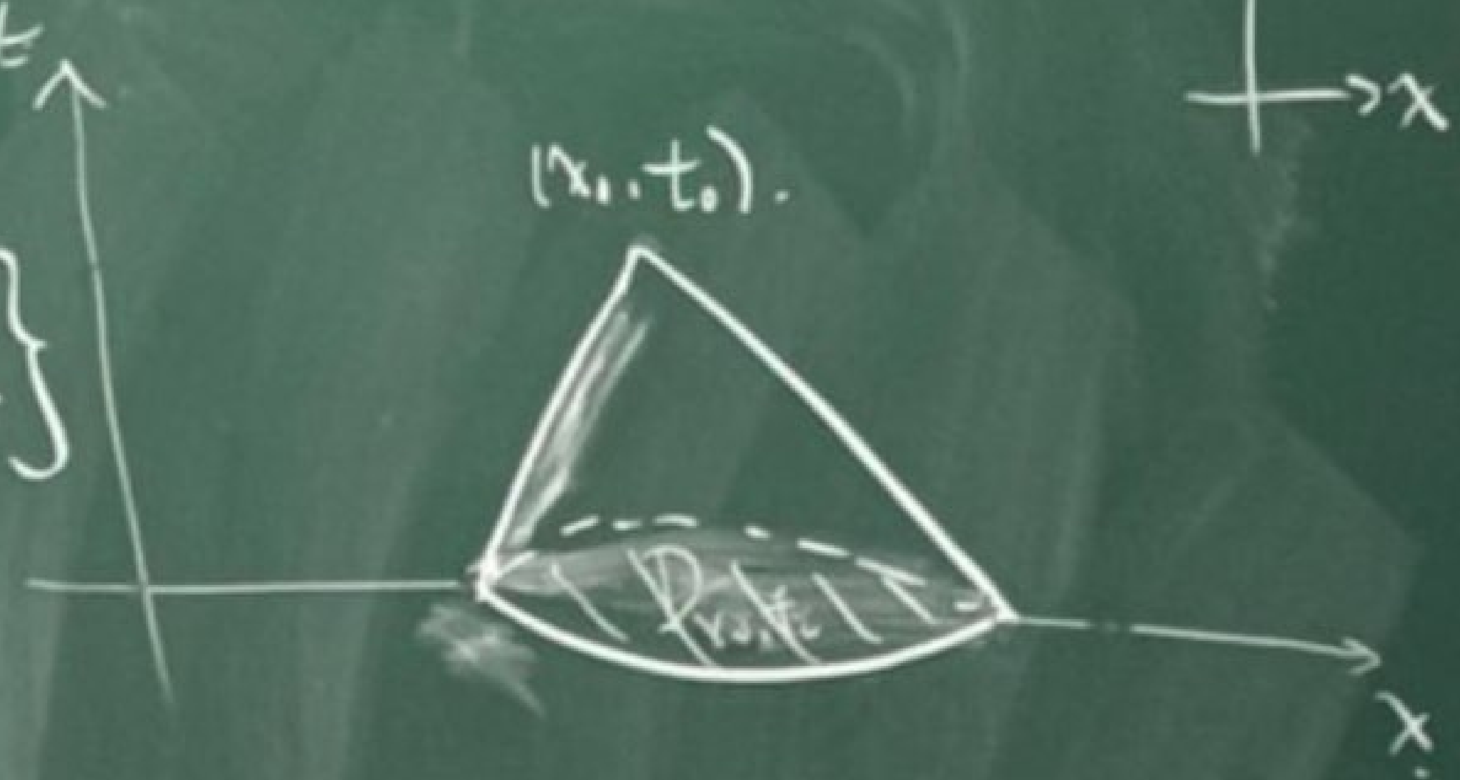
$\boxed{\text{Poisson 公式}}$

当$n=2$时，由Poisson公式，$(f\equiv 0)$

u在(x_0,t_0)的值由初值φ、ψ在
$\{x\in\mathbb{R}^2 : |x-x_0|\le t_0\}\triangleq D_{x_0,t_0}$上的值决定
而与初值在其它区域上的值无关.

当$n=3$时，由Kirchhoff公式，$(f\equiv 0)$，u在(x_0,t_0)的值由初值φ、ψ在$\{x \mid |x-x_0|=t_0\}$的值有关.
而与初值在此球面外的值无关.

任给(x_0,t_0)，称$D_{x_0,t_0}=\{x\in\mathbb{R}^n \mid |x-x_0|\le t_0\}$
为依赖区域

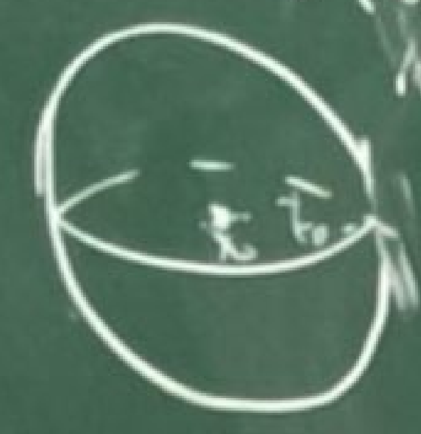


给定x_0，令$J_{x_0}=\{(x,t)\in\mathbb{R}^n\times\mathbb{R} \mid x_0\in D_{x,t}\}$ 则J_{x_0}中点的依赖区域包含x_0点.
称J_{x_0}为x_0的影响区域.
对于一个区域$D\subset\mathbb{R}^n$，$J_D=\bigcup_{x_0\in D} J_{x_0}$ 称为D的影响区域

有限传播速度

(x_0,t_0)

Huygens原理.



$|x-x_0|=t_0$　$|x-x_0|=t_0$

§4.2. 初边值问题（分离变量法）

考虑混合问题
$$\begin{cases} \partial_t^2 u - \partial_x^2 u = f(x,t) & x\in[0,l],\ t\geq 0. \\ u(x,0)=\varphi(x),\ \partial_t u(x,0)=\psi(x) \\ u(0,t)=g_1(t),\ u(l,t)=g_2(t). \end{cases}$$

1. $f\equiv 0$、$g_1(t)\equiv 0$、$g_2(t)\equiv 0$.

由Sturm-Liouville，$-\partial_x^2$ $u(0)=0$、$u(l)=0$ $\{X_n(x)\}_{n=1}^{\infty}$ 特征函数组是 $L^2([0,l])$ 中的完备正交基

将 $u(\cdot,t)$ 看作 $L^2([0,l])$ 中的函数.

则
$$u(x,t)=\sum_{n=1}^{\infty} T_n(t)\, X_n(x)$$

$v(x)\in L^2([0,l])$.
$$v(x)=\sum_{n=1}^{\infty} c_n X_n(x)$$

$u(x,t)=$