§4.2. 初边值问题 — 分离变量法

$$\begin{cases} \partial_t^2 u-\partial_x^2 u=f(x,t) \\ u(x,0)=\varphi(x),\ \partial_t u(x,0)=\psi(x) \\ u(0,t)=g_1(t),\ u(L,t)=g_2(t) \end{cases} \qquad 0\le x\le L,\ t\ge 0$$

☆ $g_1(t)\equiv g_2(t)\equiv 0,\ f(x,t)\equiv 0.$

$-\partial_x^2 \quad u(0)=0,\ u(L)=0.$

由 Sturm-Liouville. $\{X_n(x)\}_{n=1}^{\infty}$ 特征函数.

构成 $L^2([0,L])$ 中的完备正交基.

$$\begin{cases} -X_n''(x)=\lambda_n X_n(x), \\ X_n(0)=0,\ X_n(L)=0 \end{cases}$$

$$u(x,t)=\sum_{n=1}^{\infty} T_n(t)X_n(x). \quad (*)$$

将(*)代入 $\begin{cases} \partial_t^2 u-\partial_x^2 u=0 \\ u(x,0)=\varphi(x),\ \partial_t u(x,0)=\psi(x) \\ u(0,t)=0,\ u(L,t)=0 \end{cases}$

$$\begin{cases} \sum_{n=1}^{\infty} T_n''(t)X_n(x)-\sum_{n=1}^{\infty} T_n(t)X_n''(x)=0 \\ \sum_{n=1}^{\infty} T_n(0)X_n(x)=\varphi(x),\quad \sum_{n=0}^{\infty} T_n'(0)X_n(x)=\psi(x) \end{cases}$$

$$\Longrightarrow \begin{cases} \sum_{n=1}^{\infty}\left(T_n''(t)X_n(x)+\lambda_n T_n(t)X_n(x)\right)=0. \\ \sum_{n=1}^{\infty} T_n(0)X_n(x)=\varphi(x) \\ \sum_{n=1}^{\infty} T_n'(0)X_n(x)=\psi(x) \end{cases}$$

$$\Longrightarrow \begin{cases} T_n''(t)+\lambda_n T_n(t)=0 \\ T_n(0)=\dfrac{\int_0^L \varphi(x)X_n(x)\,dx}{\int_0^L [X_n(x)]^2\,dx}, \\ T_n'(0)=\dfrac{\int_0^L \psi(x)X_n(x)\,dx}{\int_0^L [X_n(x)]^2\,dx} \end{cases}$$

$(f,g)=\int_0^L f(x)g(x)\,dx$

$$\sum_{n=1}^{\infty}\left(T_n''(t)X_n(x)+\lambda_n T_n(t)X_n(x),\ X_{n_0}(x)\right)=0$$

$$\Longrightarrow T_{n_0}''(t)(X_{n_0},X_{n_0})+\lambda_{n_0}T_{n_0}(t)(X_{n_0},X_{n_0})=0$$

$$\Longrightarrow T_{n_0}''(t)+\lambda_{n_0}T_{n_0}(t)=0.$$

$$T_n(0)(X_n,X_n)=(\varphi,X_n).$$

Step1. 設 $u(x,t)=T(t)X(x)$. 則

$$\begin{cases} T''(t)X(x)-T(t)X''(x)=0. \\ T(t)X(0)=0,\ T(t)X(L)=0. \end{cases}$$

$$\begin{cases} \dfrac{T''(t)}{T(t)}=\dfrac{X''(x)}{X(x)} \triangleq -\lambda. \\ X(0)=0,\ X(L)=0. \end{cases}$$

$$\Rightarrow \begin{cases} X''(x)+\lambda X(x)=0 \\ X(0)=0,\ X(L)=0. \end{cases}$$

若 $\lambda<0$, $X(x)=C_1e^{\sqrt{-\lambda}x}+C_2e^{-\sqrt{-\lambda}x}$

$$\begin{cases} X(0)=C_1+C_2=0, \\ X(L)=C_1e^{\sqrt{-\lambda}L}+C_2e^{-\sqrt{-\lambda}L}=0 \end{cases} \Rightarrow C_1=C_2=0.$$

若 $\lambda=0$. 則 $X(x)=C_1x+C_2$ $\quad X(0)=C_2=0.$

$X(L)=C_1L=0 \Rightarrow C_1=0.$

若 $\lambda>0$. $X(x)=C_1\cos(\sqrt{\lambda}x)+C_2\sin(\sqrt{\lambda}x)$

$X(0)=C_1=0$

$X(L)=C_2\sin(\sqrt{\lambda}L)=0.$

$\Rightarrow \sqrt{\lambda}L=n\pi,\ n=1,2,\cdots$

$\Rightarrow \lambda_n=\left(\frac{n\pi}{L}\right)^2$

$X_n(x)=\sin\left(\frac{n\pi}{L}x\right),\ n=1,2,\cdots$

Step 2. 令 $\varphi_n=\frac{2}{L}\int_0^L\varphi(x)\sin\left(\frac{n\pi}{L}x\right)dx,$

$\psi_n=\frac{2}{L}\int_0^L\psi(x)\sin\left(\frac{n\pi}{L}x\right)dx.$

則 $$\begin{cases} T_n''(t)+\left(\frac{n\pi}{L}\right)^2T_n(t)=0. \\ T_n(0)=\varphi_n,\ T_n'(0)=\psi_n \end{cases}$$

設 $T_n(t)=A_n\cos\left(\frac{n\pi}{L}t\right)+B_n\sin\left(\frac{n\pi}{L}t\right)$

則 $T_n(0)=A_n=\varphi_n,\quad T_n'(0)=\frac{n\pi}{L}B_n=\psi_n.$

$\Rightarrow T_n(t)=\varphi_n\cos\left(\frac{n\pi}{L}t\right)+\frac{L}{n\pi}\psi_n\sin\left(\frac{n\pi}{L}t\right).$

$$u(x,t)=\sum_{n=1}^{\infty}\left(\varphi_n\cos\left(\frac{n\pi}{L}t\right)+\frac{L}{n\pi}\psi_n\sin\left(\frac{n\pi}{L}t\right)\right)\sin\left(\frac{n\pi}{L}x\right).$$

是原方程的解.

☆ $f(x,t)$ 不恒为零. $g_1(t)=g_2(t)\equiv 0$.

$$\Rightarrow \left(T_n''(t)+\lambda_n T_n(t)\right)(X_n, X_n)=(f, X_n).$$

$$\Rightarrow \begin{cases} T_n''(t)+\lambda_n T_n(t)=\dfrac{(f, X_n)}{(X_n, X_n)} \triangleq \underline{\underline{f_n(t)}}. \\ T_n(0)=\varphi_n,\quad T_n'(0)=\psi_n. \end{cases}$$

$u(x,t)=\sum_{n=1}^{\infty} T_n(t) X_n(x)$ 是 u 的解。

☆ $f(x,t)$、$g_1(t)$、$g_2(t)$ 都不恒为零.

令 $v(x,t)=u(x,t)-\left(\frac{L-x}{L}g_1(t)+\frac{x}{L}g_2(t)\right)$.

则 $v(0,t)=u(0,t)-g_1(t)=0$

$v(L,t)=u(L,t)-g_2(t)=0$

$$\begin{cases} \partial_t^2 v-\partial_x^2 v=f(x,t)-\left(\frac{L-x}{L}g_1''(t)+\frac{x}{L}g_2''(t)\right). \\ v(x,0)=\varphi(x)-\left(\frac{L-x}{L}g_1(0)+\frac{x}{L}g_2(0)\right),\quad \partial_t v(0)=\psi(x)-\left(\frac{L-x}{L}g_1'(0)+\frac{x}{L}g_2'(0)\right). \\ v(0,t)=0,\quad v(L,t)=0 \end{cases}$$

例. 解方程 $\begin{cases} \partial_t u-\partial_x^2 u=0 & 0\le x\le L,\quad t\ge 0 \\ u(x,0)=\varphi(x) \\ u(0,t)=0,\quad u_x(L,t)+hu(L,t)=0,\quad h>0. \end{cases}$

解: Step 1. 求特征函数.

设 $u(x,t)=T(t)X(x)$. 则 $\begin{cases} T'(t)X(x)-T(t)X''(x)=0 \\ T(t)X(0)=0,\quad T(t)X'(L)+hT(t)X(L)=0. \end{cases}$

$$\Rightarrow \begin{cases} \dfrac{T'(t)}{T(t)}=\dfrac{X''(x)}{X(x)}=-\lambda \\ X(0)=0,\quad X'(L)+hX(L)=0. \end{cases}$$

$$\begin{cases} X''(x)+\lambda X(x)=0. \\ X(0)=0,\quad X'(L)+hX(L)=0 \end{cases}$$

① 若 $\lambda<0$. 则 $X(x)=C_1 e^{\sqrt{-\lambda}x}+C_2 e^{-\sqrt{-\lambda}x}$.

$X(0)=C_1+C_2=0 \Rightarrow C_2=-C_1 \Rightarrow X(x)=C_1\left(e^{\sqrt{-\lambda}x}-e^{-\sqrt{-\lambda}x}\right)$.

$X'(L)+hX(L)=C_1\left(\sqrt{-\lambda}e^{\sqrt{-\lambda}L}+\sqrt{-\lambda}e^{-\sqrt{-\lambda}L}\right)$

$\Rightarrow C_1=0 \qquad +C_1 h\left(e^{\sqrt{-\lambda}L}-e^{-\sqrt{-\lambda}L}\right)=0.$

② $\lambda=0$. $X(x)=C_1x+C_2$.

$X(0)=C_2=0 \Rightarrow X(x)=C_1x$.

$X'(l)+hX(l)=C_1+hC_1l=C_1(1+hl)=0 \Rightarrow C_1=0$

③ $\lambda>0$.

$X(x)=C_1\cos(\sqrt{\lambda}x)+C_2\sin(\sqrt{\lambda}x)$.

$X(0)=C_1=0 \Rightarrow X(x)=\sin(\sqrt{\lambda}x)$.

$\boxed{\tan x=-\frac{x}{hl}}$

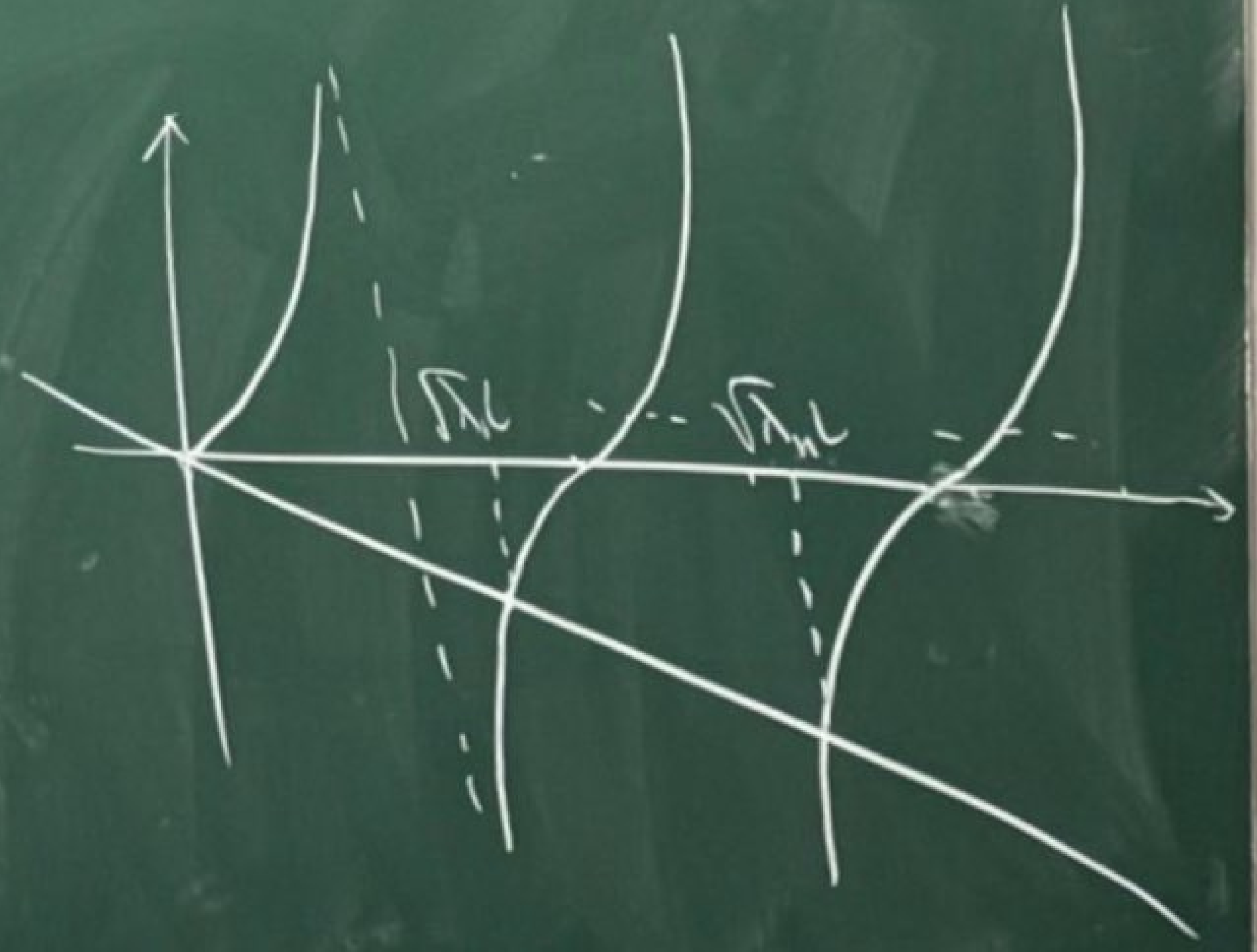
$\tan(\sqrt{\lambda_n}l)=-\frac{\sqrt{\lambda_n}l}{hl}=-\frac{\sqrt{\lambda_n}}{h}$

$X'(l)+hX(l)=\sqrt{\lambda}\cos(\sqrt{\lambda}l)+h\sin(\sqrt{\lambda}l)=0$.

$\Rightarrow \tan(\sqrt{\lambda}l)=-\frac{\sqrt{\lambda}l}{hl}$

$\exists\, \lambda_1<\cdots<\lambda_n<\cdots$, 使得

$\tan(\sqrt{\lambda_n}l)=-\frac{\sqrt{\lambda_n}}{h}$



Step 2. $\frac{T_n'(t)}{T_n(t)}=-\lambda_n$. $\Rightarrow \begin{cases} T_n'(t)+\lambda_nT_n(t)=0 \\ T_n(0)=\frac{(\varphi,X_n)}{(X_n,X_n)}\triangleq\varphi_n. \end{cases}$ $\Rightarrow T_n(t)=e^{-\lambda_nt}\varphi_n$.

$\sum_{n=1}^{\infty}T_n(0)X_n(x)=\varphi(x)=\sum_{n=1}^{\infty}\varphi_nX_n(x)$

$T_n(0)(X_n,X_n)=(\varphi,X_n)$.

原方程的解为

$u(x,t)=\sum_{n=1}^{\infty}e^{-\lambda_nt}\varphi_n\sin(\sqrt{\lambda_n}x)$.

$u(x,t)=\sum_{n=1}^{\infty}T_n(t)X_n(x)$.

$\sum_{n=1}^{\infty}T_n'(t)X_n(x)-\sum_{n=1}^{\infty}T_n(t)X_n''(x)=0$.

$\sum_{n=1}^{\infty}(T_n'(t)+\lambda_nT_n(t))X_n(x)=0$.

例. 令 $B=\{(x,y) \mid x^2+y^2<1\}$. 考虑 B 上的方程

$$\begin{cases} \Delta u = 0 & \text{in } B. \\ u = \varphi & \text{on } \partial B. \end{cases}$$

用极坐标 (r,θ)

$$\begin{cases} \partial_r^2 u + \frac{1}{r}\partial_r u + \frac{1}{r^2}\partial_\theta^2 u = 0. \\ u(r,\theta)\big|_{r=1} = \varphi(\cos\theta, \sin\theta). \end{cases}$$

令 $u(r,\theta) = R(r)\Theta(\theta)$.

$$R''(r)\Theta(\theta) + \frac{1}{r}R'(r)\Theta(\theta) + \frac{1}{r^2}R(r)\Theta''(\theta) = 0.$$

$$\Longrightarrow \left(R''(r) + \frac{1}{r}R'(r)\right)\Theta(\theta) + \frac{R(r)}{r^2}\Theta''(\theta) = 0.$$

$$\Longrightarrow \frac{r^2R''(r) + rR'(r)}{R(r)} = -\frac{\Theta''(\theta)}{\Theta(\theta)} \triangleq \lambda.$$

$$\begin{cases} \Theta''(\theta) + \lambda\Theta(\theta) = 0 \\ \Theta(\theta+2\pi) = \Theta(\theta) \end{cases}$$