分离变量法

例. 令 $B=\{(x,y)\mid x^2+y^2<1\}$. 考虑 B 上的

Laplace 方程 $\begin{cases}\Delta_{xy} u=0 & \text{in } B\\ u=\varphi & \text{on } \partial B\end{cases}$

解. 用极坐标 (r,θ).

$$\partial_r^2 u+\frac{1}{r}\partial_r u+\frac{1}{r^2}\partial_\theta^2 u=0$$

令 $u(r,\theta)=R(r)\Theta(\theta)$.

$$\partial_t^2 u-\partial_x^2 u=f \qquad 0\le x\le l,\ t\ge 0$$

$\longrightarrow u(x,0)=\varphi(x)\ \partial_t u(x,0)=\psi(x)$

$u(0,t)=0 \quad u(l,t)=0$

$$u(x,t)=T(t)X(x)$$

$$\frac{T''(t)}{T(t)}=\frac{X''(x)}{X(x)}=-\lambda$$

$X(0)=0,\ X(l)=0.$

Step 1 $\{X_n(x)\}_{n=1}^{\infty}$

Step 2. $u(x,t)=\sum_{n=1}^{\infty}T_n(t)X_n(x)$.

$T_n(t)$

则 $R''(r)\Theta(\theta)+\frac{1}{r}R'(r)\Theta(\theta)+\frac{1}{r^2}R(r)\Theta''(\theta)=0.$

$$\Longrightarrow \frac{R''(r)+\frac{1}{r}R'(r)}{R(r)}+\frac{1}{r^2}\frac{\Theta''(\theta)}{\Theta(\theta)}=0$$

$$\Longrightarrow \frac{r^2R''(r)+rR'(r)}{R(r)}=-\frac{\Theta''(\theta)}{\Theta(\theta)}\triangleq\lambda.$$

由边界条件 $u(r\cos\theta, r\sin\theta)\big|_{r=1}=\varphi(r\cos\theta, r\sin\theta)\big|_{r=1}=\varphi(\cos\theta,\sin\theta)$.

$\underbrace{R(r)\Theta(\theta)}$

$$\begin{cases}\Theta''(\theta)+\lambda\Theta(\theta)=0.\\ \Theta(\theta)=\Theta(\theta+2\pi)\end{cases}$$

若 $\lambda<0$. 则 $\Theta(\theta)=C_1e^{-\sqrt{-\lambda}\theta}+C_2e^{\sqrt{-\lambda}\theta} \Rightarrow \Theta(\theta)\equiv 0.$

若 $\lambda=0$. 则 $\Theta(\theta)=C_1\theta+C_2$ 由 $\Theta(\theta)=\Theta(\theta+2\pi)$,

$\Longrightarrow C_1\theta+C_2=C_1(\theta+2\pi)+C_2 \Rightarrow C_1=0.$

$\Theta_0(\theta)=1.$

若$\lambda>0$，则 $\Theta(\theta)=C_1\cos(\sqrt{\lambda}\theta)+C_2\sin(\sqrt{\lambda}\theta)$.

$\sqrt{\lambda}\,2\pi=2k\pi$. $\quad \cos(\sqrt{\lambda}\theta)=\cos(\sqrt{\lambda}(\theta+2\pi))$

$\Longrightarrow \lambda_k=k^2,\ k=1,2,\cdots \quad \Longrightarrow \sqrt{\lambda}\cdot 2\pi=2k\pi$.

$\Theta_k(\theta)=C_1\cos(k\theta)+C_2\sin(k\theta)$.

再求$R(r)$： $r^2R''(r)+rR'(r)-k^2R(r)=0$.

令$r=e^t$，则 $\partial_t^2R-k^2R=0$.

若$k\neq 0$，则 $R(e^t)=D_1e^{kt}+D_2e^{-kt}$.

$\Longrightarrow R(r)=D_1r^k+\underline{D_2r^{-k}}$ 舍掉

$R_k(r)=r^k,\ k\neq 0$.

若$k=0$，则 $\partial_t^2R=0 \Longrightarrow R(e^t)=D_1t+D_2$

$\Longrightarrow R(r)=D_1\ln r+D_2$ 舍掉.

$R_0(r)=D_0$

于是 $u(x)=D_0+\sum_{k=1}^{\infty}r^k(C_k\cos(k\theta)+D_k\sin(k\theta))$

由边条件，$u\big|_{\partial B}=D_0+\sum_{k=1}^{\infty}(C_k\cos(k\theta)+D_k\sin(k\theta))=\varphi(\cos\theta,\sin\theta)\triangleq\tilde{\varphi}(\theta)$.

$D_0=\int_0^{2\pi}\tilde{\varphi}(\theta)d\theta,\quad C_k\int_0^{2\pi}\cos^2(k\theta)\,d\theta=\int_0^{2\pi}\tilde{\varphi}(\theta)\cos(k\theta)\,d\theta \Longrightarrow C_k=----.\quad k=1,2,\cdots$

$D_k\int_0^{2\pi}\sin^2(k\theta)d\theta=\int_0^{2\pi}\tilde{\varphi}(\theta)\sin(k\theta)d\theta \Longrightarrow D_k=----.\quad k=1,2,\cdots$

$$
\begin{cases}
u_{tt} - u_{xx} = f(x,t) & 0<x<l,\ t>0 \\
u(x,0) = \varphi(x),\ \partial_t u(x,0) = \psi(x), & 0 \le x \le l \\
(-u_x + \alpha u)\big|_{x=0} = g_1(t) & \\
(u_x + \beta u)\big|_{x=l} = g_2(t) & ,\ \alpha,\beta > 0
\end{cases}
$$

令 $v(x) = u(x) - \left(C_1 x^2 g_1(t) + C_2 (l-x)^2 g_2(t)\right)$

$$(-v_x + \alpha v)\big|_{x=0} = g_1(t) - \left(2C_2 l(-1) g_2(t) + \alpha C_2 l^2 g_2(t)\right) = 0$$

$$v_x + \beta v = g_2(t) - \left(2C_1 l g_1(t) + \beta C_1 l^2 g_1(t)\right) = 0$$

$$\Rightarrow g_2(t) = C_1(2l + \beta l^2) g_1(t)$$

$$= \underbrace{C_1(2l+\beta l^2)\, C_2(-2l + \alpha l^2)}\, g_2(t)$$

取 $C_1 = \dfrac{1}{\alpha l^2 - 2l}$，$C_2 = \dfrac{1}{2l + \beta l^2}$ 则

$$v(x) = u(x) - \left(\frac{1}{\alpha l^2 - 2l} x^2 g_1(t) + \frac{(l-x)^2}{2l+\beta l^2} g_2(t)\right)$$

则
$$
\begin{cases}
v_{tt} - v_{xx} = f(x,t) - \left(\dfrac{1}{\alpha l - 2l^2} x^2 g_1''(t) + \dfrac{1}{2l+\beta l^2}(l-x)^2 g_2''(t) - \dfrac{2}{\alpha l - 2l^2} g_1(t) - \dfrac{2}{2l+\beta l^2} g_2(t)\right) \triangleq F(x,t) \\
v(x,0) = \varphi(x) - \left(\dfrac{1}{\alpha l^2 - 2l} x^2 g_1(0) + \dfrac{1}{2l+\beta l^2}(l-x)^2 g_2(0)\right) \triangleq \Phi(x) \\
\partial_t v(x,0) = \psi(x) - \left(\dfrac{1}{\alpha l^2 - 2l} x^2 g_1'(0) + \dfrac{1}{2l+\beta l^2}(l-x)^2 g_2'(0)\right) \triangleq \Psi(x) \\
(-v_x + \alpha v)\big|_{x=0} = 0 \\
(v_x + \beta v)\big|_{x=l} = 0
\end{cases}
$$

设 $u(x,t)=T(t)X(x)$. 则

$$\begin{cases} T''(t)X(x)-T(t)X''(x)=0 \\ T(t)\left(-X'(x)+\alpha X(x)\right)\Big|_{x=0}=0 \\ T(t)\left(X'(x)+\beta X(x)\right)\Big|_{x=l}=0 \end{cases} \Longrightarrow \frac{T''(t)}{T(t)}=\frac{X''(x)}{X(x)}\triangleq-\lambda$$

$$\begin{cases} X''(x)+\lambda X(x)=0 \\ -X'(0)+\alpha X(0)=0 \\ X'(l)+\beta X(l)=0 \end{cases}$$

若 $\lambda<0$, $X(x)=C_1e^{-\sqrt{-\lambda}x}+C_2e^{\sqrt{-\lambda}x}$

$$\begin{cases} -C_1(-\sqrt{-\lambda})-C_2(\sqrt{-\lambda})+\alpha(C_1+C_2)=0 \\ C_1(-\sqrt{-\lambda})e^{-\sqrt{-\lambda}l}+C_2\sqrt{-\lambda}e^{\sqrt{-\lambda}l}+\beta\left(C_1e^{-\sqrt{-\lambda}l}+C_2e^{\sqrt{-\lambda}l}\right)=0 \end{cases}$$

$$\Longrightarrow C_1=C_2=0$$

若 $\lambda=0$. $X(x)=C_1x+C_2$

$$\begin{cases} -C_1+\alpha C_2=0 \\ C_1+\beta(C_1l+C_2)=0 \end{cases} \Longrightarrow C_1=C_2=0.$$

若 $\lambda>0$, 则 $X(x)=C_1\cos(\sqrt{\lambda}x)+C_2\sin(\sqrt{\lambda}x)$

$X'(x)=-C_1\sqrt{\lambda}\sin(\sqrt{\lambda}x)+C_2\sqrt{\lambda}\cos(\sqrt{\lambda}x)$

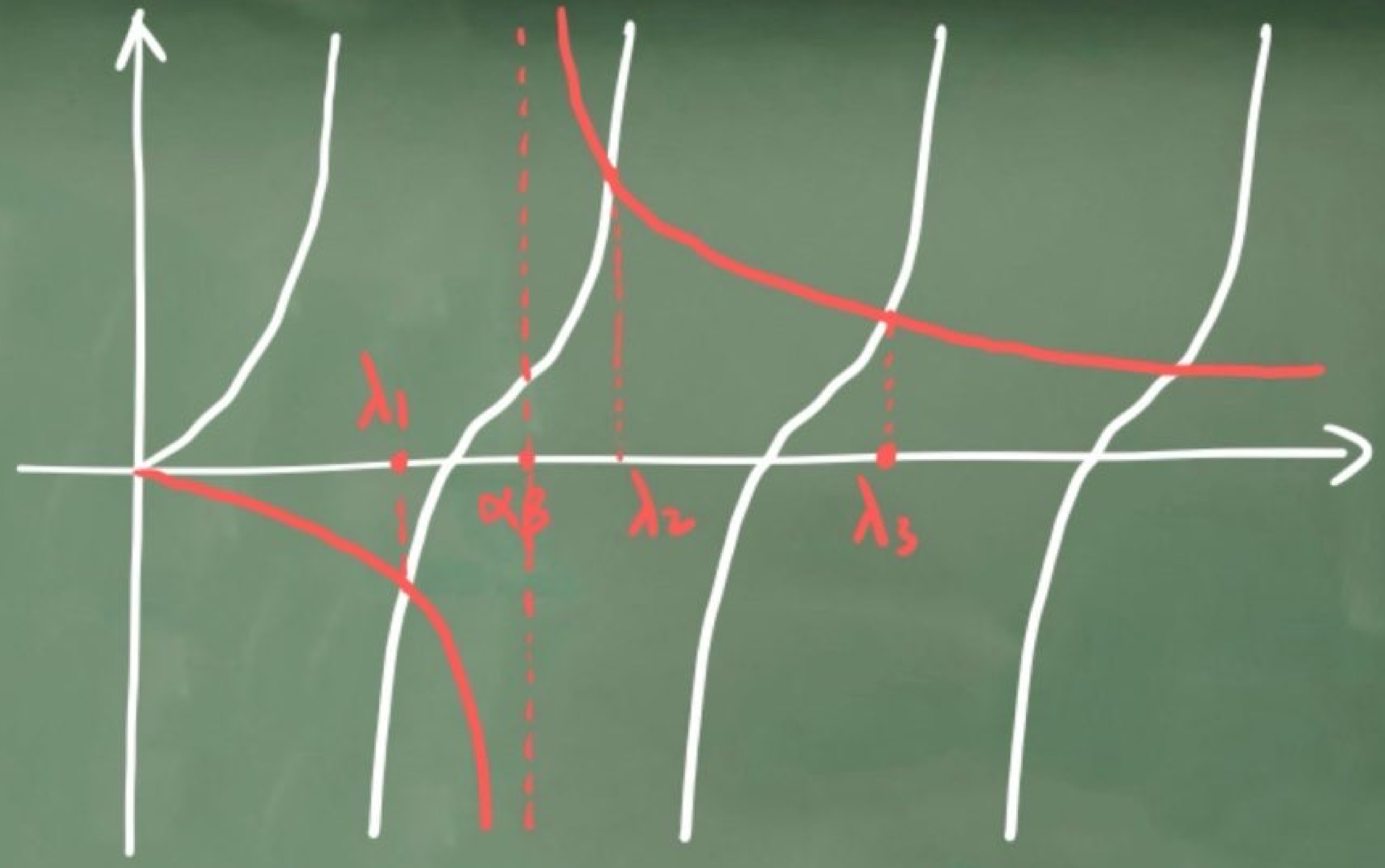
$-C_2\sqrt{\lambda}+\alpha C_1=0$

$$-C_1\sqrt{\lambda}\sin(\sqrt{\lambda}l)+\underbrace{C_2\sqrt{\lambda}}_{\alpha C_1}\cos(\sqrt{\lambda}l)+\beta C_1\cos(\sqrt{\lambda}l)+\beta\underbrace{C_2}_{\frac{\alpha C_1}{\sqrt{\lambda}}}\sin(\sqrt{\lambda}l)=0$$

$$\Longrightarrow \left(-C_1\sqrt{\lambda}+\frac{\beta\alpha C_1}{\sqrt{\lambda}}\right)\sin(\sqrt{\lambda}l)+(\alpha C_1+\beta C_1)\cos(\sqrt{\lambda}l)=0.$$

$$\Longrightarrow \tan(\sqrt{\lambda}l)=-\frac{\alpha+\beta}{-\sqrt{\lambda}+\frac{\alpha\beta}{\sqrt{\lambda}}}=\frac{\sqrt{\lambda}(\alpha+\beta)}{\lambda-\alpha\beta}$$

$\exists\,\lambda_n$, $n=1,2,\cdots$ 使得上式成立.



$$X_n(x) = C_1\cos(\sqrt{\lambda_n}x) + C_2\sin(\sqrt{\lambda_n}x).$$

令 $v(x,t) = \sum_{n=1}^{\infty} T_n(t)X_n(x)$. 代入

$$\sum_{n=1}^{\infty}\left(T_n''(t)X_n(x) + \lambda_n T_n(t)X_n(x)\right) = F(x,t). \qquad X_n'' = -\lambda_n X_n$$

$$\Longrightarrow \begin{cases} T_n''(t) + \lambda_n T_n = F_n(t). \\ T_n(0) = \Phi_n,\ T_n'(0) = \Psi_n, \end{cases} \quad F_n(t) = \frac{\int_0^l F(x,t)X_n(x)\,dx}{\int_0^l |X_n(x)|^2\,dx}. \quad \Phi_n = \frac{\int_0^l \Phi(x)X_n(x)\,dx}{\int_0^l |X_n(x)|^2\,dx},$$

$$\Psi_n = \frac{\int_0^l \Psi(x)X_n(x)\,dx}{\int_0^l |X_n(x)|^2\,dx}.$$

$$T_n(t) = \Phi_n\cos(\sqrt{\lambda_n}t) + \Psi_n\frac{\sin(\sqrt{\lambda_n}t)}{\sqrt{\lambda_n}} + \int_0^t \frac{\sin((t-s)\sqrt{\lambda_n})}{\sqrt{\lambda_n}}F_n(s)\,ds.$$

$$\Longrightarrow v(x,t) =$$

$$\Longrightarrow u(x,t) = \cdots$$

能量估计

考虑波动方程
$$\begin{cases} \partial_t^2 u - \Delta u = f(x,t) \\ u(x,0) = \varphi(x), \quad \partial_t u(x,0) = \psi(x) \\ u\big|_{\partial\Omega} = 0 \end{cases}$$

$x \in \Omega \subset \mathbb{R}^n$ bdd, $t \geq 0$

$\int_\Omega \operatorname{div} \vec{F}\, dx = \int_{\partial\Omega} \vec{F} \cdot n\, dS$

$\vec{F} = (0, \cdots 0, uv, \cdots 0)$

$\partial_x(uv) - u_x v$

$\int \partial_x u v\, dx$

$= \int_{\partial\Omega} uv \nu^i\, dS - \int u v_{x_i}$

方程两边同乘 $\partial_t u$.

$\partial_t u(\partial_t^2 u - \Delta u) = \partial_t u f$

$\underline{\partial_t} \frac{1}{2}(\partial_t u)^2 - \operatorname{div}(\partial_t u \nabla u) + \underline{\partial_t} \frac{1}{2}|\nabla u|^2 = \partial_t u f.$

$e(u) = \frac{1}{2}(\partial_t u)^2 + \frac{1}{2}|\nabla u|^2$

能量密度

$\partial_t \square$

乘子法.

$\underbrace{a\nabla u + b u_t + c\, x \cdot \nabla u}$

$\sum_{i=1}^n \partial_t u\, \partial_{x_i}^2 u$

$= \partial_{x_i}(\partial_t u\, \partial_{x_i} u) - \partial_t \partial_{x_i} u\, \partial_{x_i} u = \underline{\partial_{x_i}(\partial_t u\, \partial_{x_i} u)} - \partial_t \frac{1}{2}(\partial_{x_i} u)^2$

$\operatorname{grad} u = \nabla u = \begin{pmatrix} u_{x_1} \\ \vdots \\ u_{x_n} \end{pmatrix}$.

$\partial_t u \nabla u = \begin{pmatrix} \partial_t u\, u_{x_1} \\ \cdots \\ \partial_t u\, u_{x_n} \end{pmatrix}$.

$|\nabla u|^2 = u_{x_1}^2 + \cdots + u_{x_n}^2$

$\operatorname{div} \vec{F} = \frac{\partial F_1}{\partial x_1} + \cdots + \frac{\partial F_n}{\partial x_n}$

$u(x,t) \longmapsto u(x, t+t_0)$.

$\boxed{\partial_t u}$

$\lim_{t_0 \to 0} \frac{u(x,t+t_0) - u(x,t)}{t_0} = (\partial_t u)(x,t)$