能量估计

$$\begin{cases} \partial_t^2 u - \Delta u = 0 & x \in \mathbb{R}^n,\ t \geq 0. \\ u(x,0) = \varphi(x), & \partial_t u(x,0) = \psi(x) \end{cases}$$

$$\partial_t u\left(\partial_t^2 u - \Delta u\right) = 0 \iff \partial_t \underbrace{\left(\tfrac{1}{2}(\partial_t u)^2 + \tfrac{1}{2}|\nabla u|^2\right)}_{e(t)\ \text{能量密度}} = \mathrm{div}\left(\partial_t u \nabla u\right)$$

在$\mathbb{R}^n$上积分.

$$\int_{\mathbb{R}^n} \partial_t e(t)\, dx = \int_{\mathbb{R}^n} \mathrm{div}\left(\partial_t u \nabla u\right) dx = 0.$$

$$\int_{\mathbb{R}^n} \partial_t e(t)\, dx = \partial_t \int_{\mathbb{R}^n} e(t)\, dx \qquad \int_{\mathbb{R}^n} \mathrm{div}\left(\partial_t u \nabla u\right) dx = \lim_{R\to\infty} \int_{B_R(0)}$$

令 $E(t) = \int_{\mathbb{R}^n} e(t)\, dx$

$$= \frac{1}{2}\int \left((\partial_t u)^2 + |\nabla u|^2\right) dx. \quad \text{能量}$$

$$\int_\Omega \mathrm{div}\, \vec{F}\, dx = \int_{\partial\Omega} \vec{F}\cdot\vec{n}\, dS.$$

则 $\frac{d}{dt}E(t) = 0.$

故 $E(t) = E(0),\ \forall t \geq 0.$

$$\begin{cases} \partial_t^2 u - \Delta u = 0 & x \in \Omega \subset \mathbb{R}^n \text{ 有界区域} \\ u(x,0) = \varphi(x), & \partial_t u(x,0) = \psi(x) \\ u(x,t)\big|_{\partial\Omega} = 0 \end{cases}$$

方程两边同乘$\partial_t u$, 可得 $\partial_t e(t) = \mathrm{div}\left(\partial_t u \nabla u\right)$.

关于x在Ω上积分, 可得

$$\partial_t \int_\Omega e(t)\, dx = \int_\Omega \mathrm{div}\left(\partial_t u \nabla u\right) dx \overset{\text{散度定理}}{=} \int_{\partial\Omega} \partial_t u \nabla u \cdot n\, dS$$

$$= \int_{\partial\Omega} \partial_t u \frac{\partial u}{\partial n}\, dS$$

由边条件可得 $\partial_t u(x,t)\big|_{\partial\Omega} = 0.$

故 $\partial_t \int_\Omega e(t)\, dx = 0.$

令 $E(t) = \int_\Omega \frac{1}{2}\left[(\partial_t u)^2 + |\nabla u|^2\right] dx$, 则

$$E(t) \equiv E(0) = \frac{1}{2}\int_\Omega \left(|\psi(x)|^2 + |\nabla\varphi(x)|^2\right) dx,\ \forall t \geq 0.$$

$\partial_t u$
$$\begin{cases} \partial_t^2 u - \Delta u = f(x,t) & 0 \le t \le T \\ u(x,0)=\varphi,\ \partial_t u(x,0)=\psi & (W). \\ u(x,t)\big|_{\partial\Omega}=0. \end{cases}$$

方程两边同乘 $\partial_t u$，可得

$$\partial_t\left[\frac{1}{2}(\partial_t u)^2+\frac{1}{2}|\nabla u|^2\right] = \mathrm{div}(\partial_t u \nabla u) + \partial_t u f$$

在Ω上积分，可得

$$\partial_t \underbrace{\int_\Omega \left(\frac{1}{2}(\partial_t u)^2+\frac{1}{2}|\nabla u|^2\right)dx}_{E(t).} = \int_\Omega \partial_t u f\, dx$$

$$\le \frac{1}{2}\int_\Omega (\partial_t u)^2 dx + \frac{1}{2}\int_\Omega |f(x,t)|^2 dx$$

$$\le \int_\Omega \left(\frac{1}{2}(\partial_t u)^2+\frac{1}{2}|\nabla u|^2\right)dx + \frac{1}{2}\int_\Omega |f(x,t)|^2 dx$$

即 $\frac{d}{dt}E(t) \le E(t) + \frac{1}{2}\int_\Omega |f(x,t)|^2 dx.$

$$\frac{d}{dt}\left(e^{-t}E(t)\right) \le \frac{1}{2}e^{-t}\int_\Omega |f(x,t)|^2 dx$$

对t从0到t积分，可得

$$e^{-t}E(t) - E(0) \le \frac{1}{2}\int_0^t e^{-s}\int_\Omega |f(x,s)|^2 dx\, ds$$

故 $E(t) \le e^t E(0) + \frac{1}{2}e^t\int_0^t\int_\Omega |f(x,s)|^2 dx\, ds$

$$\le C_T\Big(\underbrace{E(0)}_{\frac{1}{2}\int_\Omega(|\nabla\varphi|^2+|\psi|^2)dx} + \frac{1}{2}\int_0^T\int_\Omega |f(x,t)|^2 dx\, dt\Big)$$

令 $E_0(t) = \int_\Omega |u(x,t)|^2 dx.$

则 $\frac{d}{dt}E_0(t) = 2\int_\Omega u\, u_t\, dx$

故 $\frac{d}{dt}E_0(t) \le \int_\Omega u^2 dx + \int_\Omega u_t^2 dx.$

$$= E_0(t) + \int_\Omega u_t^2 dx.$$

再由 Gronwall，

$$E_0(t) \le C_T\Big(E_0(0) + \frac{1}{2}\int_0^T \underbrace{\int_\Omega |u_t|^2 dx}\, dt\Big).$$

$$\le C_T\Big(E_0(0) + \frac{1}{2}C_T T\, C_T\big(E(0) + \frac{1}{2}\int_0^T\int_\Omega |f(x,t)|^2 dx\, dt\big)\Big).$$

$$\le C_T\Big(E_0(0) + E(0) + \int_0^T\int_\Omega |f(x,t)|^2 dx\, dt\Big),\quad \forall\, 0\le t\le T$$

推论：(W)的古典解是唯一的.

证：设 $u_1(x,t)$, $u_2(x,t)$ 是(W)的两个不同的解.

令 $v(x,t)=u_1(x,t)-u_2(x,t)$. 则

$$\begin{cases}\partial_t^2 v-\Delta v=0\\ v(x,0)=0,\ \partial_t v(x,0)=0\\ v(x,t)\big|_{\partial\Omega}=0\end{cases}$$

$\varphi_1(x)$ $\quad\varphi_2(x)$

$\psi_1(x)$ $\quad\psi_2(x)$

$\int|\nabla(\varphi_1(x)-\varphi_2(x))|^2dx<\delta$

$\int|\psi_1(x)-\psi_2(x)|^2dx<\delta$, $\iint_0|f_1-f_2|^2dxdt<\delta$.

由能量估计(☆1).

$0\le\frac{1}{2}\int_\Omega\left((\partial_t v)^2+|\nabla v|^2\right)dx\le 0$ $\Longrightarrow$ v在Ω上是常数

再由(☆2),

$\int_\Omega|v(x,t)|^2dx=0$. 故 $v(x,t)\equiv 0$ $\forall x\in\Omega$, $0\le t\le T$.

即 $u_1(x,t)\equiv u_2(x,t)$.

$$\begin{cases}\partial_t^2 u-\Delta u=0 & x\in\mathbb{R}^n\\ u(x,0)=\varphi(x),\ \partial_t u(x,0)=\psi(x)\end{cases}$$

用能量解释有限传播速度

$$\partial_t\frac{1}{2}\left[(\partial_t u)^2+|\nabla u|^2\right]=\mathrm{div}(\partial_t u\nabla u).$$

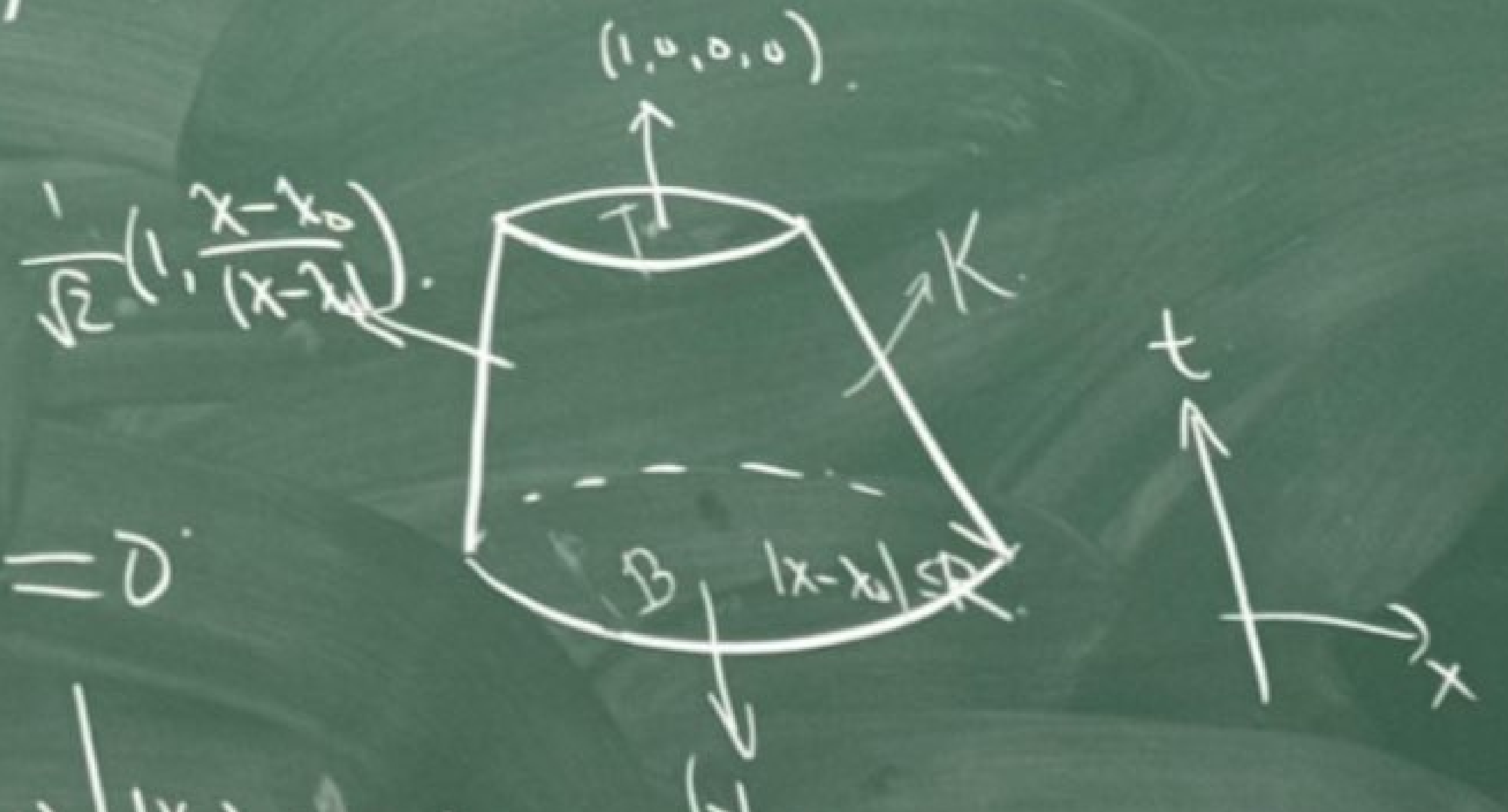
在锥台上积分可得

$$\int_{\triangle}\underbrace{\left[\partial_t e(t)-\mathrm{div}(\partial_t u\nabla u)\right]}_{\mathrm{div}_{t,x}(e(u),\,-\partial_t u\nabla u)}dxdt=0$$

由散度定理,

$$\int_{\partial\triangle}\left(e(t),\underline{-\partial_t u\nabla u}\right)\cdot\vec{n}\,dS=0.$$

$$\Longrightarrow -\int_B e(t)dx+\int_T e(t)dx+\frac{1}{\sqrt{2}}\int_K\left(e(t)-\partial_t u\frac{x-x_0}{|x-x_0|}\cdot\nabla u\right)dS=0$$

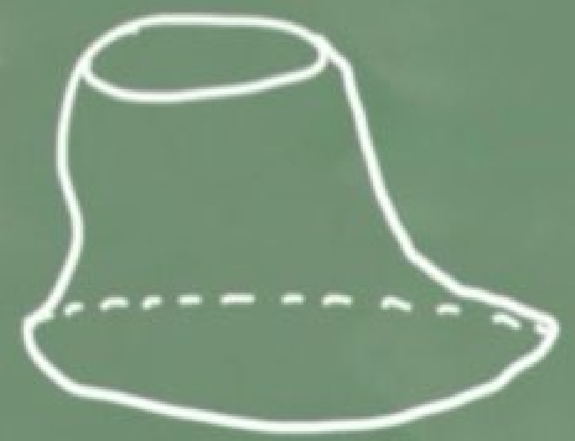


$K=\{(x,t)\mid |x-x_0|=R-t,\ 0\le t\le T\}$.

$$\int_B e(t)\,dx = \int_T e(t)\,dx + \frac{1}{2\sqrt{2}}\underbrace{\int_K \left[(\partial_t u)^2 + |\nabla u|^2 - 2\partial_t u\,\frac{x-x_0}{|x-x_0|}\nabla u\right] dS}_{\text{Flux}(0,t)}$$

$$\Vert$$

$$\underbrace{\left(\partial_t u - \frac{x-x_0}{|x-x_0|}\cdot\nabla u\right)^2}_{\geq 0} \underbrace{- \left|\frac{x-x_0}{|x-x_0|}\cdot\nabla u\right|^2 + |\nabla u|^2}_{\geq 0}$$



第二章 位势方程

我们考虑位势方程 $-\Delta u = f$ (2.1). $x\in\Omega\subset\mathbb{R}^n$, $n=2,3$.

其中 $u:\Omega\to\mathbb{R}$ 是未知函数, $f:\Omega\to\mathbb{R}$ 已知.

若 $f\equiv 0$, 则称(2.1)为Laplace方程.

边值条件: $u|_{\partial\Omega}=\varphi(x)$. Dirichlet边值

$\frac{\partial u}{\partial n}\Big|_{\partial\Omega}=\varphi(x)$ Neumann边值

$\left(\frac{\partial u}{\partial n}+\sigma u\right)\Big|_{\partial\Omega}=\varphi(x)$. Robin边值

§2.1. 调和函数

若 $\Delta u=0$ in $\mathbb{R}^n$ 即 u 是 $\mathbb{R}^n$ 上的调和函数.

则(1) $u(x+x_0)$ 也是调和的, $\forall x_0\in\mathbb{R}^n$.

(2). $u(\lambda x)$ 也是调和的, $\forall \lambda>0$.

(3) $u(Ox)$ 也是调和的 O 是旋转

如果函数 $u:\Omega\to\mathbb{R}$ 二阶连续可微，且 $\Delta u=0$ 在 Ω 上成立

则称 u 是 Ω 上的调和函数。

定义：若 $u\in C(\Omega)$，

(1). 称 u 满足平均值性质，如果 $\forall B_r(x)\subset\Omega$，$u(x)=\frac{1}{|B_r(x)|}\int_{B_r(x)}u(y)\,dy$.

(2). 称 u 满足第二平均值性质，如果 $\forall B_r(x)\subset\Omega$，

$$u(x)=\frac{1}{|\partial B_r(x)|}\int_{\partial B_r(x)}u(y)\,dS(y).$$

极坐标 $\int_{\mathbb{R}^n}f(x)dx=\int_0^{+\infty}\int_{\partial B(x_0,r)}f(y)\,dS(y)\,dr$

$$\frac{d}{dr}\int_{B(x_0,r)}f(x)\,dx=\int_{\partial B(x_0,r)}f(y)\,dS(y).$$

$n=3$. $|B_r(x)|=\frac{4}{3}\pi r^3$ $n=2$.

$|\partial B_r(x)|=4\pi r^2$

Claim: (1) $\Longleftrightarrow$ (2).

(1) $\Longrightarrow$ (2). 由于 $u(x)\frac{4}{3}\pi r^3=\int_{B_r(x)}u(y)\,dy$

两边对 r 求导，可得 $4\pi r^2u(x)=\int_{\partial B_r(x)}u(y)\,dS(y)$.

即 $u(x)=\frac{1}{|\partial B_r(x)|}\int_{\partial B_r(x)}u(y)\,dS(y)$.

(2) $\Longrightarrow$ (1). 由 (2)，$\int_{\partial B_r(x)}u(y)\,dS(y)=4\pi r^2u(x)$.

$$\int_{B_r(x)}u(y)\,dy=\int_0^r\int_{\partial B_\rho(x)}u(y)\,dS(y)\,d\rho$$

$$=\int_0^r4\pi\rho^2u(x)\,d\rho=\frac{4}{3}\pi r^3u(x).$$

即 $u(x)=\frac{1}{|B_r(x)|}\int_{B_r(x)}u(y)\,dy$.

定理2.2.(平均值公式) 若$u\in C^2(\Omega)$是Ω上的调和函数，则对任意的闭球$\overline{B_r(x)}\subset\Omega$，有

$$u(x)=\frac{1}{|\partial B_r(x)|}\int_{\partial B_r(x)}u(y)\,dS(y)$$

$$0=\int_{B_r(x)}(\Delta u)(y)\,dy=\int_{B_r(x)}\operatorname{div}\nabla u(y)\,dy\overset{\operatorname{div}}{=}\int_{\partial B_r(x)}\nabla u(y)\cdot\frac{y-x}{|y-x|}\,dS(y)\qquad \Delta u=0,\ \text{in }\Omega$$

$$\Delta u = \operatorname{div}\nabla u$$

$$=r^2\cdot\int_{|w|=1}\underline{(\nabla u)(x+rw)\cdot w}\,dS(w)=r^2\frac{d}{dr}\boxed{\int_{|w|=1}\left(u(x+rw)\right)dS(w)}$$