第二章 位势方程

§2.1 调和函数

$\Omega\subset\mathbb{R}^n$. $n=2,3$ 有界区域. $\partial\Omega\in C^\infty$

$u\in C^2(\Omega)$ $\Delta u=0,\ \forall x\in\Omega$

平均值性质: ① $\forall B_r(x)\subset\Omega$. $u(x)=\frac{1}{|B_r(x)|}\int_{B_r(x)}u(y)dy$

$u\in C(\Omega)$. ② $\forall B_r(x)\subset\Omega$, $u(x)=\frac{1}{|\partial B_r(x)|}\int_{\partial B_r(x)}u(y)dS(y)$

①⟺②

定理2.2. 若 $u\in C^2(\Omega)$ 是 Ω 上的调和函数

则对于任意的球 $B_r(x)\subset\Omega$,

$$u(x)=\frac{1}{|\partial B_r(x)|}\int_{\partial B_r(x)}u(y)dS(y).$$

证明: 由于 $\Delta u=0$ in Ω, $\forall B_r(x)\subset\Omega$.

$$0=\int_{B_r(x)}(\Delta u)(y)dy=\int_{B_r(x)}\mathrm{div}\nabla u(y)dy\ \xlongequal[\text{(散度定理)}]{\mathrm{div}}\ \int_{|y-x|=r}\nabla u(y)\cdot\frac{y-x}{r}dS(y)$$

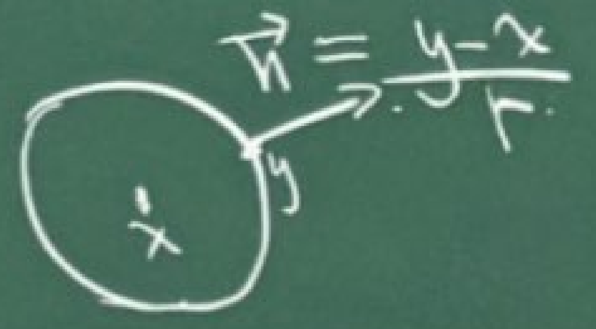
$|rw|=r$

$dS(y)=r^2dS(w)$

$$\xlongequal{y-x=rw}\int_{|w|=1}(\nabla u)(x+rw)\cdot w\,r^2dS(w).$$

$$=r^2\int_{|w|=1}\frac{d}{dr}\big(u(x+rw)\big)dS(w).$$

$$=r^2\frac{d}{dr}\underbrace{\int_{|w|=1}u(x+rw)dS(w)}$$



$|y-x|=r$.

$$\Longrightarrow \int_{|w|=1} u(x+rw)\,dS(w) = \int_{|w|=1} u(x)\,dS(w) = u(x)\,4\pi$$

$$\Longrightarrow u(x) = \frac{1}{4\pi}\int_{|w|=1} u(x+rw)\,dS(w)$$

$$\overset{y=x+rw}{=\!=} \frac{1}{4\pi r^2}\int_{\partial B_r(x)} u(y)\,dS(y)$$

$$= \frac{1}{|\partial B_r(x)|}\int_{\partial B_r(x)} u(y)\,dS(y).$$

$\forall B_r(x)\subset\Omega$.

$$\int_{B_r(x)} f(y)\,dy = 0$$

$$\Longrightarrow f\equiv 0 \quad \text{in } \Omega.$$

定理2.3. 假设 $u\in C^2(\Omega)$ 满足对任意的 $B_r(x)\subset\Omega$,

$$u(x)=\frac{1}{|\partial B_r(x)|}\int_{\partial B_r(x)} u(y)\,dS(y)$$

则 u 是 Ω 上的调和函数. $\Delta u(x)=0,\ \forall x\in\Omega$

证明: 第一步. $\forall B_r(x)\subset\Omega$,

$$\int_{B_r(x)}(\Delta u)(y)\,dy = r^2\frac{d}{dr}\int_{|w|=1} u(x+rw)\,dS(w)$$

由平均值性质.

$$u(x)=\frac{1}{4\pi r^2}\int_{|y-x|=r} u(y)\,dS(y)$$

$$\overset{y=x+rw}{=\!=} \frac{1}{4\pi}\int_{|w|=1} u(x+rw)\,dS(w).$$

故

$$\int_{B_r(x)}(\Delta u)(y)\,dy = r^2\frac{d}{dr}\left(4\pi u(x)\right) = 0 \quad (*)$$

Step 2. 要证 $\Delta u(x)=0$. $\forall x\in\Omega$

否则. $\exists\, x_0\in\Omega$, s.t. $(\Delta u)(x_0)>c>0$.

由于 $u\in C^2(\Omega)$, 存在 $r_0>0$, s.t. $(\Delta u)(x)\ge\frac{c}{2}$, $\forall x\in B_{r_0}(x_0)$

故

$$\int_{B_{r_0}(x_0)}(\Delta u)(y)\,dy \ge \frac{c}{2}\,|B_{r_0}(x_0)|>0.$$

这与(*)矛盾! 故 $\Delta u\equiv 0$ in Ω.

定理2.3'. 假设 $u\in C(\Omega)$ 满足平均值性质

$$\forall B_r(x)\subset\Omega,\quad u(x)=\frac{1}{|\partial B_r(x)|}\int_{\partial B_r(x)}u(y)\,dS(y).$$

则 $\underline{u}$ 在 Ω 上光滑的，且 u 在 Ω 上调和.

证明：令 $\varphi\in C_0^\infty(B_1(0))$, ($\varphi\in C^\infty$, $\operatorname{supp}\varphi\subset B_1(0)$).

$$\varphi\equiv 1 \text{ on } B_{\frac12}(0),\quad \int_{R^n}\varphi(x)\,dx=1,\quad \varphi\ge 0,\quad \text{radial } (\varphi(x)=\varphi(|x|))$$

则 $$1=\int_{R^n}\varphi(x)\,dx\overset{r=|x|}{=}\int_0^\infty\int_{|w|=1}\varphi(rw)\,r^2\,dS(w)\,dr$$

$$=\int_0^\infty\int_{|w|=1}\varphi(r)\,r^2\,dS(w)\,dr$$

$$=4\pi\int_0^\infty\varphi(r)\,r^2\,dr$$

令 $\varphi_\varepsilon(x)=\frac{1}{\varepsilon^n}\varphi\left(\frac{x}{\varepsilon}\right)$, 则 $\operatorname{supp}\varphi_\varepsilon\subset B_\varepsilon(0)$.

($\varepsilon>0$ 充分小).

$$\int_{R^n}\varphi_\varepsilon(x)\,dx=\int_{R^n}\frac{1}{\varepsilon^n}\varphi\left(\frac{x}{\varepsilon}\right)dx\overset{y=\frac{x}{\varepsilon}}{=}\frac{1}{\varepsilon^n}\int_{R^n}\varphi(y)\,\varepsilon^n\,dy=\int_{R^n}\varphi(y)\,dy=1.$$

claim: $u(x)=(u*\varphi_\varepsilon)(x)$, $\forall x\in\Omega$, ε 充分小.

事实上，
$$\int_{R^n}\varphi_\varepsilon(x-y)u(y)\,dy=\int_{B_\varepsilon(x)}\varphi_\varepsilon(x-y)u(y)\,dy=\int_{|y-x|<\varepsilon}\frac{1}{\varepsilon^n}\varphi\left(\frac{y-x}{\varepsilon}\right)u(y)\,dy$$

$$\overset{\frac{y-x}{\varepsilon}=z,\ \frac{1}{\varepsilon^n}dy=dz}{=}\int_{|z|<1}\frac{1}{\varepsilon^n}\varphi(z)\,u(x+\varepsilon z)\,\varepsilon^n\,dz\overset{|y-x|<\varepsilon}{=}\int_{|z|<1}\varphi(z)\,u(x+\varepsilon z)\,dz\overset{\operatorname{supp}\varphi\subset B_1(0)}{=}\int_{R^n}\varphi(z)\,u(x+\varepsilon z)\,dz$$

用极坐标 $z=rw$

$$=\int_0^\infty\int_{|w|=1}\varphi(r)\,u(x+\varepsilon rw)\,r^2\,dS(w)\,dr.$$

$$=\int_0^\infty\varphi(r)\,r^2\underbrace{\int_{|w|=1}u(x+\varepsilon rw)\,dS(w)}\,dr.$$

由平均值性质，$u(x) = \frac{1}{4\pi r^2}\int_{\partial B_r(x)} u(y)\,dS(y)$

$\overset{y=x+r\omega}{=\!=\!=} \frac{1}{4\pi}\int_{|\omega|=1} u(x+r\omega)\,dS(\omega)$

故 $\int \varphi_\varepsilon(x-y)\,u(y)\,dy = 4\pi\, u(x)\int_0^{+\infty}\varphi(r)\,r^2\,dr = u(x)$.

因此，u是光滑的. 再由定理2.3，u是调和的.

定理2.4（Harnack 不等式）. 对于Ω上的任何连通紧子集V，存在一个仅与距离函数 $d(V,\partial\Omega)=\min\limits_{x\in V,\, y\in\partial\Omega}|x-y|$ 和维数n有关的正常数C，使得 $\sup\limits_{V} u \le C \inf\limits_{V} u$，其中$u$是$\Omega$上的任意非负调和函数. 特别地，$\forall x,y\in V$，$\frac{1}{C}u(y)\le u(x)\le C\,u(y)$.

证：要证 $\sup\limits_{V} u \le C\inf\limits_{V} u$

只要证明 $\forall x,y\in V$，$u(x)\le C\,u(y)$

Step 1. $|x-y|<r$. 则 $B_r(x)\subset B_{2r}(y)\subset\Omega$

（取 $r<\frac{1}{4}d(V,\partial\Omega)$）. 因为$u$是调和的，$u$满足平均值性质

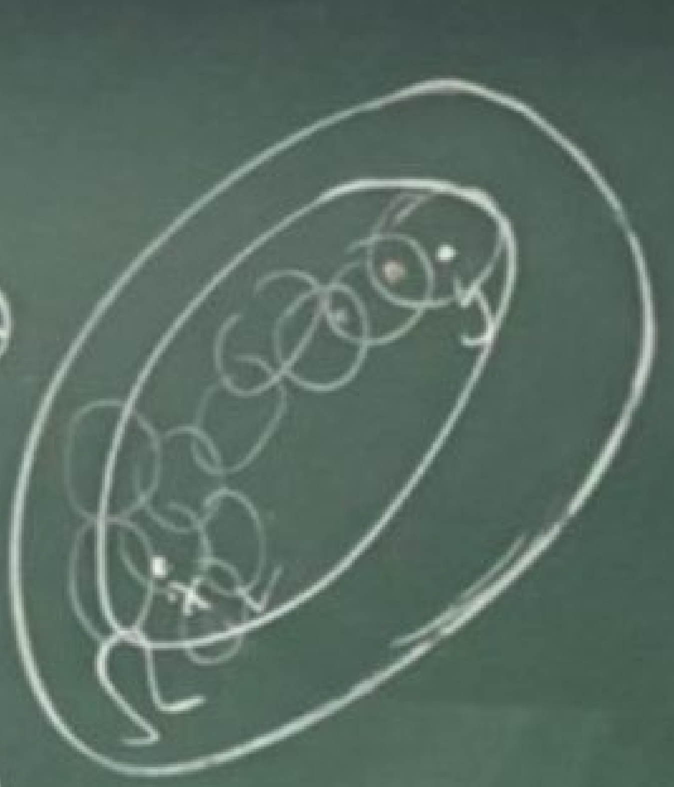
$$u(y)=\frac{1}{|B_{2r}(y)|}\int_{B_{2r}(y)}u(z)\,dz = \frac{1}{\frac{4}{3}\pi(2r)^3}\int_{B_{2r}(y)}u(z)\,dz \overset{u\ge 0}{\ge} \frac{1}{\frac{4\times 8}{3}\pi r^3}\int_{B_r(x)}u(z)\,dz = \frac{1}{2^3}u(x)$$

Step 2. $\forall x,y\in V$.

由于V是紧致、连通的

存在有限个球 $B_r(x_i)$，$i=1,\cdots,N$. $x,y\in\bigcup\limits_{i=1}^{N} B_r(x_i)$

且 $B_r(x_i)\cap B_r(x_{i+1})\ne\emptyset$



重复利用Step 1，$u(y)\ge\frac{1}{2^{nN}}u(x)$.

故 $u(x) \le 2^{nN} u(y)$

$\Longrightarrow \sup u \le 2^{nN} \inf u$.

定理2.7. 若 $u \in C(\overline{B_R})$ 是调和的 $(B_R = B_R(x_0))$

则 $|\nabla u(x_0)| \le \frac{n}{R} \max_{\overline{B_R}} u$

(梯度估计).

证: 由于 u 是调和的, 故 u 满足平均值性质.

再由定理2.3, u 是光滑的. 于是在 $\Delta u = 0$ 两边

同时作用 ∂_{x_i}, 可得 $\Delta \partial_{x_i} u = 0$.

再由平均值性质,

$$\partial_{x_i} u(x) = \frac{1}{|B_R(x_0)|} \int_{B_R(x_0)} \partial_{x_i} u(y)\, dy = \frac{1}{|B_R(x_0)|} \int_{B_R(x_0)} \mathrm{div}(0, \cdots, 0, u, 0, \cdots, 0)\, dy$$

散度定理 $\frac{1}{\frac{4}{3}\pi R^3} \int_{\partial B_R(x_0)} u \nu^i \, dS(y)$.

$$\Longrightarrow |\partial_{x_i} u(x)| \le \frac{1}{\frac{4}{3}\pi R^3} \int_{\partial B_R(x_0)} |u| \, dS(y)$$

$$\le \frac{1}{\frac{4}{3}\pi R^3} \max_{B_R(x_0)} |u| \cdot 4\pi R^2 = \frac{3}{R} \max_{B_R} |u|.$$

定理2.8.（Liouville 定理）假设 u 是 $\mathbb{R}^n$ 上的有界调和函数，则 u 是常数.

证明：设 $|u(x)| \le M$, $\forall x \in \mathbb{R}^n$. 由于 u 在 $\mathbb{R}^n$ 上调和，故 $\forall R > 0$，u 在 B_R 上调和，且 $u \in C(\overline{B_R})$.

由梯度估计，$|\nabla u(x)| \le \frac{n}{R} \max\limits_{\mathbb{R}^n} |u| \le \frac{nM}{R}$，$\forall x$.

令 $R \to +\infty$，则 $|\nabla u(x)| = 0$. 故 u 是常数.