$$\begin{cases} \Delta u = f & x\in\Omega \\ u = \varphi & x\in\partial\Omega \end{cases}$$

§2.2 基本解与 Green 函数

目标：求解位势方程 $\Delta u = f$. (*)

若 $\Delta\Gamma = \delta$，则 $f = f*\delta = f*(\Delta\Gamma) = \Delta(f*\Gamma)$

(Dirac 函数) 分布

令 $u = \Gamma * f$，则 $\Delta u = f \quad \forall x\in\mathbb{R}^n$

claim: f radial $\xRightarrow{\text{若(*)的解唯一}}$ (*) 的解 u 也是径向的

$\forall O \in SO(n)$，

$\Delta(u(Ox)) = (\Delta u)(Ox) = f(Ox) = f(x)$.

由解的唯一性，$u(Ox) = u(x)$. $\Rightarrow u$ 是径向的.

$u(Ox) = u(x)$，$\forall O$ 旋转 $SO(n)$

由 δ 是径向的，故 Γ 是径向函数，即 $\Gamma(r)$.

用极坐标 (r,ω)，则

$$\partial_r^2\Gamma + \frac{n-1}{r}\partial_r\Gamma = 0. \quad r>0.$$

令 $v = \partial_r\Gamma$，则 $\partial_r v + \frac{n-1}{r}v = 0 \Rightarrow v(r) = C_1 r^{-(n-1)}$

$$\Rightarrow \Gamma(r) = \begin{cases} C_1\ln r + C & n=2 \\ \frac{C_1}{2-n} r^{-(n-2)} + C & n\geq 3 \end{cases}, \quad C_1, C \text{ 是任意常数}.$$

$\Delta\Gamma = C_1\Delta\ln r$

令 $\Gamma(r) = \begin{cases} \frac{1}{2\pi}\ln r & n=2 \\ -\frac{1}{4\pi}\frac{1}{r} & n=3 \end{cases}$

则 $\Delta\Gamma = \delta$

Green 公式：若 $u, v \in C^2(\Omega)\cap C^1(\bar\Omega)$，则

$$\int_\Omega u\Delta v\,dx = \int_\Omega [\operatorname{div}(u\nabla v) - \nabla u\cdot\nabla v]\,dx = \int_{\partial\Omega} u\frac{\partial v}{\partial n}\,dS - \int_\Omega \nabla u\cdot\nabla v\,dx.$$

$u\partial_{x_i}^2 v = \partial_{x_i}(u\partial_{x_i}v) - \partial_{x_i}u\,\partial_{x_i}v.$

且 $\int_\Omega v\Delta u\,dx = \int_{\partial\Omega} v\frac{\partial u}{\partial n}dS - \int_\Omega \nabla u\cdot\nabla v\,dx$. (2).

(1)−(2) 可得 $\int_\Omega (u\Delta v - v\Delta u)dx = \int_{\partial\Omega}\left(u\frac{\partial v}{\partial n} - v\frac{\partial u}{\partial n}\right)dS$.

(Green 第二公式)

Claim: 若 ($n=3$) $u\in C^2(\Omega)\cap C^1(\bar\Omega)$ 满足 $\Delta u=0$ in Ω, 则 $\forall x_0\in\Omega$,

$$u(x_0)=\int_{\partial\Omega}\left[-\frac{1}{4\pi}u\frac{\partial}{\partial n}\left(\frac{1}{|x-x_0|}\right)+\frac{1}{4\pi|x-x_0|}\frac{\partial u}{\partial n}\right]dS.$$

$0\in\Omega-\{x_0\}$

只须证明 $0\in\tilde\Omega=\Omega-\{x_0\}$.

$$u(0)=\int_{\partial\tilde\Omega}\left[-\frac{1}{4\pi}u\frac{\partial}{\partial n}\left(\frac{1}{|x|}\right)+\frac{1}{4\pi|x|}\frac{\partial u}{\partial n}\right]dS \quad (\star)$$

若 $(\star)$ 已证, 令 $v(x)=u(x+x_0)$.

则 $v\in C^2(\tilde\Omega)\cap C^1(\bar{\tilde\Omega})$

且 $\Delta v=(\Delta u)(x+x_0)=0$. $(x+x_0\in\Omega)$

$\forall x\in\tilde\Omega$.

于是
$$\underset{u(x_0)}{\overset{\parallel}{v(0)}}=\int_{\partial\tilde\Omega}\left[-\frac{1}{4\pi}v(x)\frac{\partial}{\partial n}\left(\frac{1}{|x|}\right)+\frac{1}{4\pi|x|}\frac{\partial v}{\partial n}\right]dS$$

$$\parallel$$

$$\int_{\partial\tilde\Omega}\left[-\frac{1}{4\pi}u(x+x_0)\frac{\partial}{\partial n}\left(\frac{1}{|x|}\right)+\frac{1}{4\pi|x|}\frac{\partial u}{\partial n}(x+x_0)\right]dS$$

$$\overset{y=x+x_0}{=\!=}\int_{\partial\Omega}-\frac{1}{4\pi}u(y)\frac{\partial}{\partial n}\left(\frac{1}{|y-x_0|}\right)+\frac{1}{4\pi|y-x_0|}\frac{\partial u}{\partial n}(y)\,dS$$

$x\in\tilde\Omega=\Omega-\{x_0\}$

$y=x+x_0\in\Omega-\{x_0\}+x_0\in\Omega$

若 $0\in\Omega$，则

$$u(0)=\underbrace{\int_{\partial\Omega}\left[-\frac{1}{4\pi}u(x)\frac{\partial}{\partial n}\left(\frac{1}{|x|}\right)+\frac{1}{4\pi|x|}\frac{\partial u}{\partial n}\right]dS}_{\int_{\partial\Omega}\left[u(x)\frac{\partial}{\partial n}\left(-\frac{1}{4\pi|x|}\right)-\left(-\frac{1}{4\pi|x|}\right)\frac{\partial u}{\partial n}\right]dS}$$

令 $\Omega_\varepsilon=\Omega\setminus\overline{B_\varepsilon(0)}$，则 $u(x)$ $v(x)=-\frac{1}{4\pi|x|}$ 在 Ω_ε 上是调和的。于是，在 Ω_ε 上对 u,v 应用第二Green公式，可得

$\Delta v=\delta$

$\frac{\partial}{\partial n}=-\frac{\partial}{\partial r}$

$B_\varepsilon(0)$

Ω

$$0=\int_{\partial\Omega_\varepsilon}\left(u\frac{\partial v}{\partial n}-v\frac{\partial u}{\partial n}\right)dS$$

$$=\underline{\int_{\partial\Omega}\left(u\frac{\partial v}{\partial n}-v\frac{\partial u}{\partial n}\right)dS}+\int_{|x|=\varepsilon}\Big[\underbrace{u\left(\frac{\partial}{\partial r}\left(\frac{1}{4\pi|x|}\right)\right)}_{①\ -u(0)}+\underbrace{\frac{1}{4\pi|x|}\frac{\partial u}{\partial r}}_{②}\Big]dS.$$

$$① = \int_{|x|=\varepsilon}u\left(-\frac{1}{4\pi|x|^2}\right)dS=-\underline{\frac{1}{4\pi\varepsilon^2}\int_{|x|=\varepsilon}u(x)\,dS}\xrightarrow{\varepsilon\to 0}-u(0).$$

$$\frac{1}{4\pi\varepsilon^2}\underbrace{\int_{|x|=\varepsilon}\left(u(x)-u(0)\right)dS}+\underline{\underline{u(0)}}$$

$$\le\int_{|x|=\varepsilon}\max_{B_\varepsilon}|\nabla u|\,\varepsilon\,dS$$

$$\le\max_{\Omega}|\nabla u|\,\cancel{\frac{1}{4\pi\varepsilon^2}}\cdot\varepsilon\,\cancel{4\pi\varepsilon^2}\longrightarrow 0,\ \text{as }\varepsilon\to 0.$$

$$|②|=\left|-\frac{1}{4\pi\varepsilon}\underline{\int_{|x|=\varepsilon}\frac{\partial u}{\partial r}dS}\right|\le\frac{1}{4\pi\varepsilon}\max_{\Omega}\left|\frac{\partial u}{\partial r}\right|\cdot 4\pi\varepsilon^2\longrightarrow 0,\ \text{as}\to 0.$$

$$\int_{|x|=\varepsilon}\frac{\partial u}{\partial n}dS\overset{\text{div}}{=}\int_{B_\varepsilon(0)}\Delta u\,dx=0.$$

$\frac{\partial}{\partial r}$

因此，当 $\varepsilon\to 0$ 时，有

$$u(0)=\int_{\partial\Omega}\left[-\frac{1}{4\pi}u\frac{\partial}{\partial n}\left(\frac{1}{|x|}\right)+\frac{1}{4\pi|x|}\frac{\partial u}{\partial n}\right]dS$$

如果 $\Delta u=f$

$$u(x_0)=\int_{\Omega}-\frac{1}{4\pi|x-x_0|}f(x)\,dx+\int_{\partial\Omega}\left[-\frac{1}{4\pi}u\frac{\partial}{\partial n}\left(\frac{1}{|x-x_0|}\right)+\frac{1}{4\pi|x-x_0|}\frac{\partial u}{\partial n}\right]dS.\quad (A)$$

若存在 $g(x)$ 在 Ω 上调和，且 $g|_{\partial\Omega}=\frac{1}{4\pi|x-x_0|}\Big|_{\partial\Omega}$，则

对 u,g 在 Ω 上应用第二Green公式.

$$\int_{\Omega}-g f\,dx=\int_{\partial\Omega}\left(u\frac{\partial g}{\partial n}-g\frac{\partial u}{\partial n}\right)dS=\int_{\partial\Omega}\left[u\frac{\partial g}{\partial n}-\frac{1}{4\pi|x-x_0|}\frac{\partial u}{\partial n}\right]dS.\quad (B)$$

于是 (A)+(B)，可得

$$u(x_0)=\int_{\Omega}-\frac{1}{4\pi|x-x_0|}f(x)\,dx+\int_{\partial\Omega}u\frac{\partial}{\partial n}\underbrace{\left(-\frac{1}{4\pi|x-x_0|}+g\right)}_{G(x,x_0)}dS+\int_{\Omega}f g\,dx$$

重写为

$$u(x)=\int_{\Omega}f(y)\,g^x(y)\,dy+\int_{\Omega}-\frac{1}{4\pi|x-y|}f(y)\,dy+\int_{\partial\Omega}\varphi\frac{\partial}{\partial n}\underbrace{\left(-\frac{1}{4\pi|x-y|}+g^x(y)\right)}\,dy$$

$G(x,x_0)$：格林函数

① $G(x,x_0)$ 在 Ω 上调和，除了 $x=x_0$.

② $G(x,x_0)=0$，$\forall x\in\partial\Omega$

③ $G(x,x_0)+\frac{1}{4\pi|x-x_0|}$ 在 Ω 上调和.