$$\begin{cases}\Delta u = f & \text{in } \Omega \\ u = \varphi & \text{on } \partial\Omega\end{cases} \quad (*)$$

$\Omega = R^n$ $\quad \Delta u = f$ in R^n

$$\Gamma(x) = \begin{cases}\frac{1}{2\pi}\ln|x| & n=2 \\ -\frac{1}{4\pi|x|} & n=3\end{cases} \quad \text{基本解}$$

$\Longrightarrow u(x) = (\Gamma * f)(x)$

$\Omega \subset R^n$ 有界区域.

第二Green公式. $u, v \in C^2(\Omega) \cap C^1(\bar{\Omega})$.

$$\int_\Omega (u\Delta v - v\Delta u)\,dx = \int_{\partial\Omega}\left(u\frac{\partial v}{\partial n} - v\frac{\partial u}{\partial n}\right)dS$$

$\boxed{n=3}$

$$(*) \quad u(x_0) = \int\underbrace{\left(-\frac{1}{4\pi|x-x_0|} + g^{x_0}(x)\right)}_{G(x,x_0)} f(x)\,dx + \int_{\partial\Omega} u\frac{\partial}{\partial n}\left(-\frac{1}{4\pi|x-x_0|} + g^{x_0}(x)\right)dS(x)$$

其中 $g^{x_0}(x)$ 是 Ω 上的调和函数, $g^{x_0}(x)\big|_{\partial\Omega} = \frac{1}{4\pi|x-x_0|}\Big|_{\partial\Omega}$.

令 $G(x,x_0) = -\frac{1}{4\pi|x-x_0|} + g^{x_0}(x)$, 则

$$u(x_0) = \int_\Omega G(x,x_0)f(x)\,dx + \int_{\partial\Omega} u(x)\frac{\partial G}{\partial n}(x,x_0)\,dS(x)$$

$n=2$ $\quad G(x,x_0) = \frac{1}{2\pi}\log|x| + g^{x_0}(x)$.

称 $G(x,x_0)$ 为 Ω 上关于 x_0 的 Green 函数. 它满足

(1). $G(x,x_0)$ 在 Ω 上二阶连续可微且调和. 除了 $x=x_0$

(2). $G(x,x_0)\big|_{\partial\Omega} = 0$.

(3). $G(x,x_0) + \frac{1}{4\pi}\frac{1}{|x-x_0|}$ 在 Ω 上二次连续可微且调和

性质: $G(x,x_0) = G(x_0,x)$, $\forall x_0 \in \Omega$, $x \in \Omega$.

证明: 令 $u(x) = G(x,a)$, $v(x) = G(x,b)$, $a, b \in \Omega$.

要证 $u(b) = v(a) \Longleftrightarrow G(b,a) = G(a,b)$

由于 $\Delta u = 0$, $\forall x \neq a$. $\quad$ | $\quad \Delta v = 0$, $\forall x \neq b$.

$u\big|_{\partial\Omega} = 0$ $\quad$ | $\quad v\big|_{\partial\Omega} = 0$.

$u + \frac{1}{4\pi|x-a|}$ 调和. $\quad$ | $\quad v + \frac{1}{4\pi|x-b|}$ 调和.

令 $\Omega_\varepsilon=\Omega\setminus\left(\overline{B_\varepsilon(a)}\cup\overline{B_\varepsilon(b)}\right)$ 在 Ω_ε 上对 $u(x)$, $v(x)$ 用第二Green公式，有

$$0=\int_{\Omega_\varepsilon}(u\Delta v-v\Delta u)\,dx=\int_{\partial\Omega_\varepsilon}\left(u\frac{\partial v}{\partial n}-v\frac{\partial u}{\partial n}\right)dS.$$

$$=\underbrace{\int_{\partial\Omega}\left(u\frac{\partial v}{\partial n}-v\frac{\partial u}{\partial n}\right)dS}_{=0}+\int_{|x-a|=\varepsilon}\left(u\frac{\partial v}{\partial n}-v\frac{\partial u}{\partial n}\right)dS+\int_{|x-b|=\varepsilon}\left(u\frac{\partial v}{\partial n}-v\frac{\partial u}{\partial n}\right)dS.$$

$\Omega_\varepsilon=\Omega\setminus\left(\overline{B_\varepsilon(a)}\cup\overline{B_\varepsilon(b)}\right)$

$$\int_{|x-a|=\varepsilon}\left(u\frac{\partial v}{\partial n}-v\frac{\partial u}{\partial n}\right)dS$$

$$=\int_{|x-a|=\varepsilon}\left[\left(u+\frac{1}{4\pi|x-a|}\right)\frac{\partial v}{\partial n}-v\frac{\partial}{\partial n}\left(u+\frac{1}{4\pi|x-a|}\right)\right]dS \quad ⓐ$$

$$-\int_{|x-a|=\varepsilon}\frac{1}{4\pi|x-a|}\frac{\partial v}{\partial n}\,dS \quad ⓑ$$

$$+\int_{|x-a|=\varepsilon}v\frac{\partial}{\partial n}\left(\frac{1}{4\pi|x-a|}\right)dS. \quad ⓒ$$

ⓐ：由第二Green公式，

$$ⓐ=-\int_{|x-a|=\varepsilon}\left[\left(u+\frac{1}{4\pi|x-a|}\right)\frac{\partial v}{\partial n}-v\frac{\partial}{\partial n}\left(u+\frac{1}{4\pi|x-a|}\right)\right]dS$$

$$=\int_{B_\varepsilon(a)}\left[\Delta\left(u+\frac{1}{4\pi|x-a|}\right)v-\left(u+\frac{1}{4\pi|x-a|}\right)\Delta v\right]dx.$$

$=0.$

$$ⓑ=\frac{1}{4\pi\varepsilon}\int_{|x-a|=\varepsilon}\frac{\partial v}{\partial n}\,dS=\frac{1}{4\pi\varepsilon}\int_{B_\varepsilon(a)}\Delta v\,dx=0$$

③: $\frac{\partial}{\partial n}\left(\frac{1}{4\pi|x-a|}\right) = -\frac{x-a}{|x-a|}\cdot\nabla\left(\frac{1}{4\pi|x-a|}\right) = \frac{1}{4\pi|x-a|^2}$.

$$③ = \frac{1}{4\pi\varepsilon^2}\int_{|x-a|=\varepsilon} v(x)\,dS = \underbrace{\frac{1}{4\pi\varepsilon^2}\int_{|x-a|=\varepsilon}\big(v(x)-v(a)\big)\,dS}_{(\Delta)} + v(a)$$

$-v(a)+v(a)$

$$|(\Delta)| \le \frac{1}{4\pi\varepsilon^2}\int_{|x-a|=\varepsilon}\max_{B_\varepsilon(a)}|\nabla v|\,|x-a|\,dx$$

$$\le \max_{B_\varepsilon(a)}|\nabla v|\,\varepsilon \longrightarrow 0,\ \text{as } \varepsilon\to 0.$$

故 $③ \longrightarrow v(a)$, as $\varepsilon\to 0$.

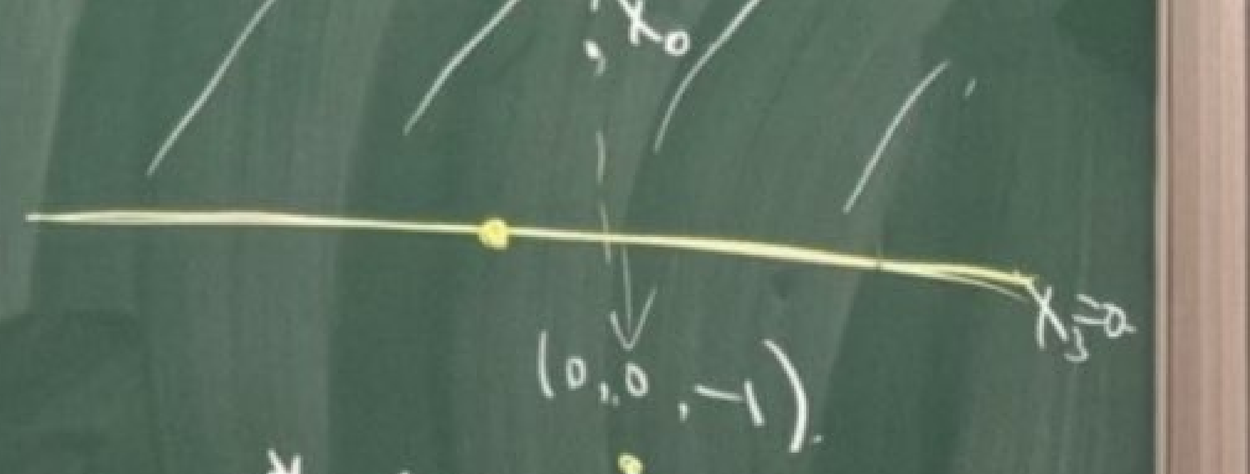
$\Delta\left(-\frac{1}{4\pi|x-x_0^*|}\right)=0$, $\forall x\ne x_0^*$, $x_0\in\mathbb{R}^3_+$.

同理, $\int_{|x-b|=\varepsilon}\left(u\frac{\partial v}{\partial n} - v\frac{\partial u}{\partial n}\right)dS \longrightarrow -u(b)$. as $\varepsilon\to 0$.

故 $0 = v(a)-u(b)$. $\Longleftrightarrow u(b)=v(a)$.

Green函数的求法:

1. 半空间 $\mathbb{R}^3_+ = \{(x_1,x_2,x_3)\mid x_3>0\}$.



由(3), $G(x,x_0) = -\frac{1}{4\pi|x-x_0|} + \frac{1}{4\pi|x-x_0^*|}$, $x_0^* = (x_0^1, x_0^2, -x_0^3)$.

2. $B_R(0)$ 球上的 Green函数.

$G(x,x_0) = -\frac{1}{4\pi|x-x_0|} + \frac{C}{4\pi|x-x_0^*|}$ $\quad x_0^*\notin B_R(0)$.

希望 $G(x,x_0)=0$, $\forall x\in\partial B_R(0)$.

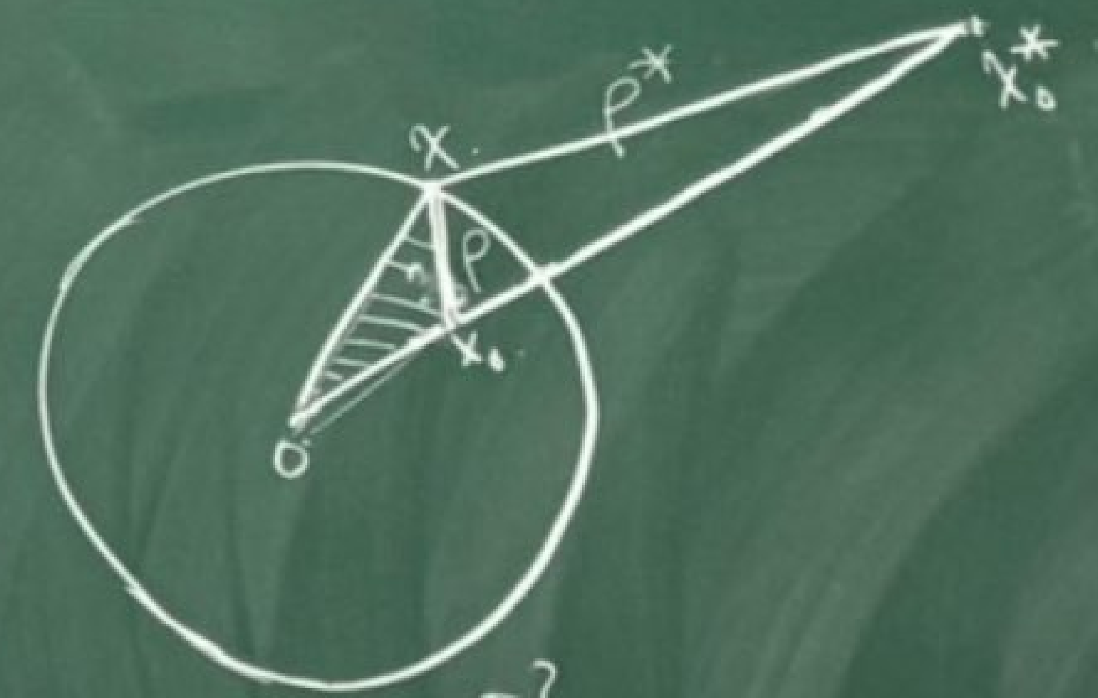
若 $\Delta_小 \sim \Delta_大$. 令 $\rho=|x-x_0|$, $\rho^*=|x-x_0^*|$.

$$\frac{|x_0|}{R} = \frac{\rho}{\rho^*} = \frac{R}{|x_0^*|} = \frac{1}{C}$$

$|x_0^*| = \frac{R^2}{|x_0|}$,

$x_0^* = \frac{R^2}{|x_0|}\cdot\frac{x_0}{|x_0|} = \frac{R^2}{|x_0|^2}x_0$.

$C = \frac{R}{|x_0|}$.



$$G(x,x_0) = -\frac{1}{4\pi|x-x_0|} + \frac{R}{|x_0|}\frac{1}{4\pi|x-x_0^*|},\quad x_0^* = \frac{R^2}{|x_0|^2}x_0.$$

由Poisson公式, $x\in\partial B_R(0)$. $\dfrac{\partial G}{\partial n} =$

$$\nabla G(x) = \frac{x-x_0}{4\pi|x-x_0|^3} - \frac{R}{|x_0|}\frac{x-x_0^*}{4\pi|x-x_0^*|^3} \qquad x\in\partial B_R(0).$$

$$= \frac{x-x_0}{4\pi|x-x_0|^3} - \frac{R}{|x_0|}\frac{|x_0|^3}{R^3}\frac{x-x_0^*}{4\pi|x-x_0|^3}$$

$$= \frac{1}{4\pi|x-x_0|^3}\left(x-x_0-\frac{|x_0|^2}{R^2}x+\underbrace{\frac{|x_0|^2}{R^2}x_0^*}_{x_0}\right).$$

$$= \frac{1}{4\pi|x-x_0|^3}\frac{R^2-|x_0|^2}{R^2}x$$

$$\frac{\partial G}{\partial n} = \frac{x}{|x|}\cdot\nabla G = \frac{R^2-|x_0|^2}{4\pi R^2|x-x_0|^3}R = \frac{R^2-|x_0|^2}{4\pi R|x-x_0|^3}$$

$$\begin{cases}\Delta u = f & x\in B_R(0)\\ u|_{\partial B_R(0)}=\varphi.\end{cases}$$

的解为

$$u(x) = \int_{B_R(0)} G(y,x)f(y)\,dy + \int_{\partial B_R(0)}\varphi(y)\frac{R^2-|x|^2}{4\pi R|y-x|^3}dS(y)$$

$$\| \quad \frac{R^2-|x|^2}{4\pi R}\int_{\partial B_R(0)}\frac{\varphi(y)}{|y-x|^3}dS(y).$$

$$\Omega_\varepsilon=\Omega\setminus\left(\overline{B_\varepsilon(a)}\cup\overline{B_\varepsilon(b)}\right)$$

定理2.4' (Harnack不等式). 设u在$B_R(x_0)$内调和且$u\geq 0$. 则

$$\frac{R}{R+r}\frac{R-r}{R+r}u(x_0)\le u(x)\le\frac{R}{R-r}\frac{R+r}{R-r}u(x_0),\quad 其中\ r=|x-x_0|<R.$$

证明: 不妨设$x_0=0$. 则只要证明 $\dfrac{R}{R+r}\dfrac{R-r}{R+r}u(0)\le u(x)\le \dfrac{R}{R-r}\dfrac{R+r}{R-r}u(0)$

由Poisson公式 $u(x)=\dfrac{R^2-|x|^2}{4\pi R}\displaystyle\int_{|y|=R}\frac{u(y)}{|x-y|^3}dS(y)$

由于 $|x|=r$, $|y|=R$, 故

$$R-r\le|x-y|\le R+r$$

因此,

$$u(x)\le\frac{R^2-r^2}{4\pi R}\int_{|y|=R}\frac{u(y)}{(R-r)^3}dy$$

$$=\frac{R+r}{4\pi R}\cdot\frac{1}{(R-r)^2}\int_{|y|=R}u(y)\,dy\cdot\frac{1}{4\pi R^2}\cdot 4\pi R^2$$

$$\overset{平均值性质}{=} \frac{R(R+r)}{(R-r)^2}u(0)$$

定理2.8 (Liouville定理). 设u是$\mathbb{R}^n$上的有上界(或有下界)的调和函数. 则u是一个常数.

证明: 设 $u(x)\le M$, $\forall x\in\mathbb{R}^n$. 令$v(x)=M-u(x)$. 则 $v(x)\ge 0$. 且 v在$\mathbb{R}^n$上调和.

于是, 由Harnack不等式,

$$\frac{R(R-r)}{(R+r)^2}v(x_0) \le v(x) \le \frac{R(R+r)}{(R+r)^2}v(x_0), \quad \forall x_0\in\mathbb{R}^n.\ R>|x-x_0|.$$

令$R\to+\infty$, 则有 $v(x_0)\le v(x)\le v(x_0)$. 即 $v(x)\equiv v(x_0)$, $\forall x\in\mathbb{R}^n$.

§2.3 极值原理与最大模估计 $u\in C^2(\Omega)\cap C(\bar\Omega)$

考虑 $-\Delta u+c(x)u=f$ (2.38) $x\in\Omega\subset\mathbb{R}^n$ 有界区域

$c(x)\ge 0$, $\forall x\in\Omega$.

定理2.21 假设 $c(x)\ge 0$, $f<0$. 如果 $u\in C^2(\Omega)\cap C(\bar\Omega)$ 满足方程(2.38). 则$u(x)$不能在Ω上达到它在$\bar\Omega$上的非负最大值. 即u在$\partial\Omega$达到它在$\bar\Omega$上的最大值.

证: 若u在$x_0\in\Omega$达到它在$\bar\Omega$上的非负最大值.

则 $(\Delta u)(x_0)\le 0$, $(\nabla u)(x_0)=0$. $u(x_0)\ge 0$.

于是, $\left.\left(-\Delta u+c(x)u\right)\right|_{x=x_0}\ge 0$.

但是 $f(x_0)<0$. 矛盾!

定理2.22. 假设 $c(x)\geq 0$, $f\leq 0$. 如果 $u\in C^2(\Omega)\cap C(\bar\Omega)$ 满足方程(2.38) 且在 Ω 存在正的最大值. 则 u 必在 $\partial\Omega$ 达到它在 $\bar\Omega$ 上的非负最大值.

且 $\max\limits_{\bar\Omega} u \leq \max\limits_{\partial\Omega} u^+$, $u^+(x) = \max\{u(x), 0\}$.

证明: 令 $w(x) = u(x) + \varepsilon v(x)$, $(\varepsilon>0)$ $L = -\Delta + c(x)$.

不妨设 $0\in\Omega$, $d = \operatorname{diam}\Omega$. 则 $\forall x\in\Omega$, $|x|\leq d$

令 $v(x) = |x|^2 - d^2$. 则 $v(x)\leq 0$.

且 $Lv = -\Delta v + c(x)v \leq -2n < 0$.

于是 $Lw = Lu + \varepsilon Lv = \underset{\leq 0}{f} + \varepsilon \underset{<0}{Lv} < 0$

由定理2.21, $\max\limits_{\bar\Omega} w \leq \max\limits_{\partial\Omega} w^+$

又 $\max\limits_{\bar\Omega} w = \max\limits_{\bar\Omega}\left(u + \varepsilon(|x|^2-d^2)\right) \geq \max\limits_{\bar\Omega} u - \varepsilon d^2$

$\max\limits_{\partial\Omega} w^+ \leq \max\limits_{\partial\Omega} u^+$

因此, $\max\limits_{\bar\Omega} u \leq \max\limits_{\partial\Omega} u^+ + \varepsilon d^2$, $\forall \varepsilon>0$.

令 $\varepsilon\to 0^+$. 则有 $\max\limits_{\bar\Omega} u \leq \max\limits_{\partial\Omega} u^+$

定理2.23 (Hopf引理). 设 B_R 是 $\mathbb{R}^n$ $(n=2,3)$ 中的一个以 R 为半径的球. 在 B_R 上, $c(x)\geq 0$. 如果 $u\in C^2(\Omega)\cap C(\bar\Omega)$ 满足 (1) $Lu = -\Delta u + c(x)u \leq 0$, $\forall x\in B_R$

(2). 存在 $x_0\in\partial B_R$, 使得 $u(x)$ 在 x_0 达到它在 $\bar B_R$ 的严格的非负最大值.

即 $u(x_0) = \max\limits_{\bar B_R} u$. 且 $u(x_0) > u(x)$, $x\in B_R$

则 $\frac{\partial u}{\partial \nu}(x_0) > 0$, ν 与 ∂B_R 在 x_0 的单位外法方向夹角小于 $\frac{\pi}{2}$.

$W_\varepsilon(x) = u(x) - u(x_0) + \varepsilon v(x)$

$\boxed{W_\varepsilon(x) \text{ 在 } x_0 \text{ 达到它}}$ 在 $\bar\Omega$ 上的严格的最大值.

$\boxed{\frac{\partial u}{\partial n}(x_0) > 0.}$

$\frac{\partial u}{\partial n}(x_0) \geq 0.$

$\frac{\partial W_\varepsilon}{\partial n}(x_0) \geq 0$

$$\Longrightarrow \frac{\partial u}{\partial n}(x_0) + \varepsilon \frac{\partial v}{\partial n}(x_0) \geq 0.$$

$$\Longrightarrow \frac{\partial u}{\partial n}(x_0) \geq -\varepsilon \frac{\partial v}{\partial n}(x_0) > 0$$

$$\underline{\underline{\frac{\partial v}{\partial n}(x_0) < 0}}$$